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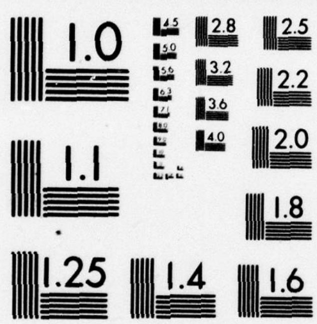

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CATASTROPHE THEORY IN THE BEHAVIORAL SCIENCES

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Catastrophe theory is a recent and rich development in mathematics. However, its possible application to the behavioral sciences is uncertain and the source of considerable controversy. The theory was reviewed with emphasis on its usefulness in the behavioral sciences. Previous attempts to apply catastrophe theory were examined and criticized. A bibliography of publications in catastrophe theory was compiled, and three new investigations were designed to test the theory's applicability. A laboratory

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study of reversible apparent movement was the most successful of these; some aspects of the data showed properties that were in accord with expectations derived from a cusp catastrophe. Computer simulations of the behavior of hypothetical neural nets revealed selected catastrophic properties and suggested possible connections between mutually inhibitory systems and the phenomena of perceptual reversals.

FOREWORD

This research was conducted in support of the In-House Independent Laboratory Research Program.

DONALD F. PARKER
Commanding Officer

SUMMARY

Problem

Catastrophe theory has been hailed as a great mathematical advance and condemned as a "mere curiosity." Its possible application to the behavioral sciences is uncertain and the source of considerable controversy. Review of the theory and new tests of its applicability are needed.

Objective

The objective of this effort was to review and analyze catastrophe theory for its possible application to the behavioral sciences, and to design and conduct new investigations bearing on its suitability as a behavioral model.

Approach

The philosophical and mathematical characteristics of catastrophe theory were studied and analyzed with emphasis on the implications of catastrophe theory for the design of psychological experiments. A bibliography of publications in catastrophe theory was compiled. Previous attempts to apply catastrophe theory to behavioral phenomena were examined and criticized. New attempts were made to apply the theory to three behavioral phenomena: the observing response in a monitoring situation, the perception of reversible depth, and the perception of reversible apparent movement. A simple but rich class of mathematical neural nets was investigated in terms of its catastrophic properties and the possible relation of these to reversible perceptions.

Findings

The review of past efforts to apply catastrophe theory revealed that there are few rigorous applications of the theory in the physical sciences, some plausible applications in the biological sciences, and some interesting metaphorical analogies to the social and behavioral sciences. In the latter cases, the analysis is typically post hoc, there are usually no supporting data, and the variables are not always clearly defined.

Three experiments were conducted in an attempt to remedy some of these weaknesses. An experiment on the observing response was not a suitable application because stable behaviors were too slow to emerge. An investigation of reversible depth was unsuccessful because perceived reversals persisted when stable perception was required as a background condition for the tests. An apparent movement study was more successful; some aspects of the data showed qualitative properties that accorded with expectations derived from a cusp catastrophe.

Computer simulations of the behavior of hypothetical neural nets revealed selected catastrophic properties and suggested possible connections between mutually inhibitory systems and the phenomena of perceptual reversals.

Conclusions

A partial success in applying catastrophe theory has been achieved, and the issues involved have been brought into clearer focus. However, there is still no convincing application of catastrophe theory to the behavioral sciences, and hence no clear resolution of the problem to which this investigation was addressed.

Recommendations

1. Efforts to find adequate behavioral applications of catastrophe theory should continue.

2. The implications of catastrophe theory for general experimental methodology should be further examined.

3. The structure of systems which might mediate overt behaviors should be examined with a view toward their possible catastrophic properties. In particular, the properties of abstract mutually inhibitory nerve nets should be explored using computer simulation techniques.

4. Published work on applications of catastrophe theory to behavioral science should be continually monitored for possible relevance to Navy problems.

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INTRODUCTION

Problem and Background

Catastrophe theory is a recent and rich development in mathematics. However, its possible application to the behavioral sciences is uncertain and the source of considerable controversy. Moreover, the substance of the theory, its alleged applications, and the surrounding critical dialogue have appeared in rather diverse forums.

The theory, which is due to the mathematician René Thom, addresses the stable or equilibrium states of systems and is particularly concerned with discontinuities in system performance that can occur in response to small continuous changes in the system's environment. Thus, many complex systems--whether they are organizational, economic, or man-machine systems—may well perform within tolerance over a wide range of their input. When "tweaked" in certain ways, however, they "break down" or exhibit unpredicted anomalies in their output, often producing a large dramatic shift in system behavior. Such discontinuities are properly called "catastrophes" in the everyday sense of the word, and they may perhaps be modeled by Thom's formal theory.

Thom's work has spurred a recent rush to "account for" alleged catastrophes in the biological, economic, ecological, and behavioral realms, but no convincing behavioral demonstrations exist to date. Thus, it is unclear whether catastrophe theory is simply a mathematical oddity that provides a loose analogy to some seeming discontinuities, or has real promise as a modeling tool whereby sudden transitions in system performance might be explained, and, even more important, anticipated or controlled.

Objective and Approach

The purpose of this effort was to review catastrophe theory, to assess its usefulness in the behavioral sciences, and to design new investigations to test its applicability.

The philosophical and mathematical characteristics of the theory were studied and analyzed with emphasis on its implications for the design of psychological experiments. A bibliography of publications in catastrophe theory was compiled. Previous attempts to apply catastrophe theory to behavioral phenomena were examined and criticized, and new attempts were made to apply the theory to three behavioral phenomena: the observing response in a monitoring situation, the perception of reversible depth, and the perception of reversible apparent movement. Finally, a simple but rich class of mathematical neural nets was investigated in terms of its catastrophic properties and the possible relation of these to reversible perceptions.

CHARACTERISTICS OF CATASTROPHE THEORY

Historical Notes and Overview

René Thom, the French mathematician, conceived catastrophe theory in its present form and is responsible for much of its development. However, the germ of the idea can be traced to earlier works by Rashevsky (1948) and Thompson (1961). While Thom makes no secret of his indebtedness to other mathematicians, he seems more enthusiastic about his predecessors in biology. Thus, Thom's motivation for the development of catastrophe theory arose from issues in biology, and his mathematical development followed from singularity theory. An independent but equivalent general theory (Thompson, 1975; Thompson & Hunt, 1975) has its roots in bifurcation theory. A compilation of publications in catastrophe theory, which includes its mathematical development and proposed applications in physics, biology, and the social and behavioral sciences, appears in the bibliography.

Controversy has surrounded catastrophe theory since the 1972 publication of Thom's book, "Stabilité structurelle et morphogénèse." For example, Zeeman (1974), who is a strong supporter of the theory, says that it "provides the deepest level of insight and lends a simplicity of understanding" and "provides a model where none was previously thought possible." On the other hand, Kolata (1977) quotes various critics who indicate that insufficient attention has been paid to scientific details of applications, that definitions are ambiguous or inconsistent, and that alleged proofs are not proofs at all. Properly speaking, the criticisms are of the applications of catastrophe theory rather than the theory itself.

The intent of the present review, then, is to examine some of the attempts to apply catastrophe theory to behavioral problems, including attempts that we ourselves have made. This goal necessitates review of some of the basic conceptual properties of catastrophe theory, and an examination of possible criteria for recognizing catastrophes in systems.

The Domain of Catastrophe Theory

A dynamic system may be described by certain quantities whose values give all necessary information about the current state of the system. Such quantities are called "state variables" by the system scientist. (Note here that these are the familiar "dependent" or "behavioral" variables of the behavioral scientist.) The goal of system science is to describe the changes in these state variables as a function of time. To accomplish this, a rule must be obtained that specifies the dependence between the state variables and time. Such a rule is called a dynamic. More precisely, then, a dynamic system consists of the state variables and the dynamic or rule that relates them in time. For example, consider a spring and its attached weight as a dynamic system. The position of the weight is the state (or behavioral) variable, and Hooke's law can be used to describe the dynamic--how the weight moves in time. Consider the national economy as a second example of a dynamic system. The state variables could be taken to be such factors as gross national product, unemployment rate, stock market prices, and inflation rate. The dynamic is the interrelationship between these variables and time as described by a particular economic theory.

In any dynamical system, particular values of the state variables are of special interest. These values are called equilibrium states and are defined as conditions for which no change with time is specified. Equilibria are divided into two classes, called stable and unstable. Stable equilibria are characterized by the property that small perturbations of the state variables away from a stable equilibrium will cause the system, through the dynamic, to restore itself to the equilibrium. (Such stable equilibria are sometimes called attractors.) All other types of equilibria are called unstable. Unstable equilibria are of little practical interest because real systems are in such states briefly, if at all. For example, a pendulum might be posed momentarily, balanced above its point of suspension, but the slightest perturbation would quickly take it away from that unstable equilibrium.

Stable equilibria, on the other hand, influence neighboring states because of the dynamic of the system. The sphere of influence of a stable equilibrium is called the basin of the equilibrium. In the first example above, the resting position of the weight at the end of the spring is a stable equilibrium. For any small displacement of the weight, the dynamic will restore the system to this resting position. The basin of this particular stable equilibrium consists of all possible positions of the weight. This is true because, from any point, the dynamic will restore the weight to its rest position.

The qualitative behavior of a dynamic system is often more important than exact quantitative results. This qualitative behavior is determined by mapping the basins of stable equilibria of the dynamic system. Once this map is made, the approximate time evolution of the system can be determined by examining the basin in which the system starts.

Thus far, our discussion has been concerned with the definition of a dynamic system and a simple classification of equilibrium states. We can now state that the domain of catastrophe theory is that class of dynamic systems whose dynamic is governed by the minimization of a real-valued function, called a potential function. This is to say that the time evolution of the system is such that the state variables seek to "achieve" the local minimum of the potential function. It then follows that the potential function determines the qualitative behavior of the system. Indeed, the local minima of the potential function correspond to the stable equilibria; and the local maxima and inflection points, to the unstable equilibria. Clearly the surrounds of the local minima will define their basins.

Time itself does not appear as a variable in the potential function. Rather, the potential function describes the long-term qualitiative behavior of the system, whether or not the dynamic of the system is known. This will obviously be useful whenever it is easier to find the potential function than to find the dynamic.

Next consider two dynamics related to two distinct potential functions on the same state variables. These represent two distinct dynamic systems. Such systems are equivalent or said to be of the same type if the qualitative behaviors of the dynamic systems are topologically equivalent. This is a mathematical characterization that roughly means that the maps of the basins of each system are approximately the same. Because the qualitative behaviors of the dynamic systems are defined by their respective potential functions,

the two dynamics are of the same type if the potential function of each can be transformed into the potential function of the other by appropriate changes in the coordinates of the state variables. These transformations of coordinates preserve the number of local minima, maxima, and inflection points of the function, and thus preserve the qualitative character of the potential function. This notion of equivalence serves to divide all dynamics into nonoverlapping classes, and leads to a definition of structural stability. A dynamic system is called structurally stable if small perturbations of its potential function produce new dynamic systems which are equivalent (in the above sense) to the original dynamic system.

In many systems, the dynamic depends on several parameters that are not themselves influenced by the dynamic system. Such parameters are called control variables in system science because the experimenter can manipulate them and thereby control the behavior of the state variables. (Note that these are the "independent variables" of the behavioral scientist.) Thus, as the control variables are changed, the quantitative behavior of the state variables changes.

If the qualitative behavior of the system also changes in response to the continuous manipulation of the control variables, the effect may be a discontinuous jump in the state variables. The realization of this discontinuity constitutes the "catastrophe." Overt catastrophic changes thus occur when an equilibrium that heretofore determined the state of the system disappears, and the system rapidly "comes under the influence" of a different attractor. In such cases, very small causes (changes in control variables) may lead to very large effects (changes in state variables).

We can now state Thom's main classification theorem of catastrophe theory: There is a class of potential functions, f, defined on n state variables and dependent on k (at most five) control variables, that has the following properties: For arbitrary points c (a k-vector) in the control space and x (an n-vector) in the state space, there are coordinates $(t_1, \ldots, t_k)$ in the control space and $(U_1, \ldots, U_n)$ in the state space such that the potential function $f(t_1, \ldots, t_k, U_1, \ldots, U_n)$ has one of the following 13 local forms:

Noncritical	U_1
Nondegenerate critical	$\pm U_1^2 \pm U_2^2 \pm \ldots \pm U_n^2$
Fold	$U_1^2 + t_1U_1 + (M)$
Cusp	$\pm (U_1^4 + t_2U_1^2 + t_1U_1) + (M)$
Swallowtail	$U_1^5 + t_3U_1^4 + t_2U_1^2 + t_1U_1 + (M)$

Butterfly	$\pm(U_1^6 + t_4U_1^4 + t_3U_1^3 + t_2U_1^2 + t_1U_1) + (M)$
Wigwam	$U_1^7 + t_5U_1^5 + t_4U_1^4 + t_3U_1^3 + t_2U_1^2 + t_1U_1 + (M)$
Elliptic umbilic	$U_1^2U_2 - U_2^3 + t_3U_1^2 + t_2U_2 + t_1U_1 + (N)$
Hyperbolic umbilic	$U_1^2U_2 + U_2^3 + t_3U_1^2 + t_2U_2 + t_1U_1 + (N)$
Parabolic umbilic	$\pm(U_1^2U_2 + U_2^4 + t_4U_2^2 + t_3U_1^2 + t_2U_2 + t_1U_1) + (N)$
Second elliptic umbilic	$U_1^2U_2 - U_2^5 + t_5U_2^3 + t_4U_2^2 + t_3U_1^2 + t_2U_2 +$ $t_1U_1 + (N)$
Second hyperbolic umbilic	$U_1^2U_2 - U_2^5 + t_5U_2^3 + t_4U_2^2 + t_3U_1^2 + t_2U_1 +$ $t_1U_1 + (N)$
Symbolic umbilic	$\pm(U_1^3 + U_2^4 + t_5U_1U_2^2 + t_4U_2^2 + t_3U_1U_2 +$ $t_2U_2 + t_1U_1 + (N)$

where $M = \pm U_2^2 \pm U_3^2 \pm \ldots \pm U_n^2$ and

$$N = \pm U_3^2 \pm U_4^2 \pm \ldots \pm U_n^2$$

Several observations are in order. First, the specific class of potential functions for which Thom's theorem is valid is really quite large. Any potential function on n state variables and at most five control variables can be approximated arbitrarily closely by a potential function in Thom's class. More important, the approximation is such that the qualitative behavior of the approximate potential function is nearly the same as that of the given potential function.

The second observation regards the coordinate systems for the control and state spaces. The theorem does not specify how to construct these coordinates; it only guarantees their existence. Note here too that any finite number of state variables is admissible, but, except for at most two of these, each appears only as a quadratic term with coefficient equal to ±1. The theorem does not specify which one or two of the state variables may take other powers or coefficients, nor does it specify the sign of the unit coefficients for

the "excess" state variables. Instead, these constructions remain as tasks for the system scientist, but again their existence is guaranteed by the theorem.

The third observation is that, of the 13 equations listed above, the last 11 exhibit catastrophes. The first two, the noncritical and the nondegenerate critical, are in fact independent of the control variables, and the qualitative behavior of the system is not affected by changes in control. Thus, no catastrophe can take place. The last 11, however, are dependent on the control variables.

Qualitative changes in any of the potential functions can be observed by differentiating each with respect to the state variables and setting the result equal to zero. The solutions to this latter equation (or equations) then define the local equilibria of the system. Now as the control variables are <u>continuously</u> changed, these equilibria may suddenly change from stable equilibria to unstable ones, or from unstable equilibria to stable ones or to normal points. Thus, the system may exhibit a catastrophe. To illustrate, take, for example, the cusp equation (with the + sign), which applies when there are two control variables and one state variable:

$$f(U_1, t_1, t_2) = U_1^{\,4} + t_2 U_1^{\,2} + t_1 U_1 + (M).$$

Differentiating <u>f</u> with respect to U_1 and equating the result to zero yields,

$$4\,U_1^3 + 2t_2\,U_1 + t_1 = 0.$$

For each fixed pair of control values, t_1 and t_2, this equation then defines the equilibrium points, U_1 stable and unstable, of the system. The surface in Figure 1 is a plot of these points; the upper and lower sheets are stable equilibria, and the fold between them represents unstable equilibria.

Thus, Figure 1 plots the "resting" values of the system for each setting of the control variables. That is, after a "long" period of time at any setting (t_1, t_2) of the control variables, the system will be near the value of the state variable (U_1) represented by the surface above (t_1, t_2). If the control variables are within the cusp-like area (under the fold in the surface), then U_1 will be near either the highest or lowest value of the three specified by the surface. Since the middle value represents an <u>unstable</u> equilibrium, the state variable will not be near that value.

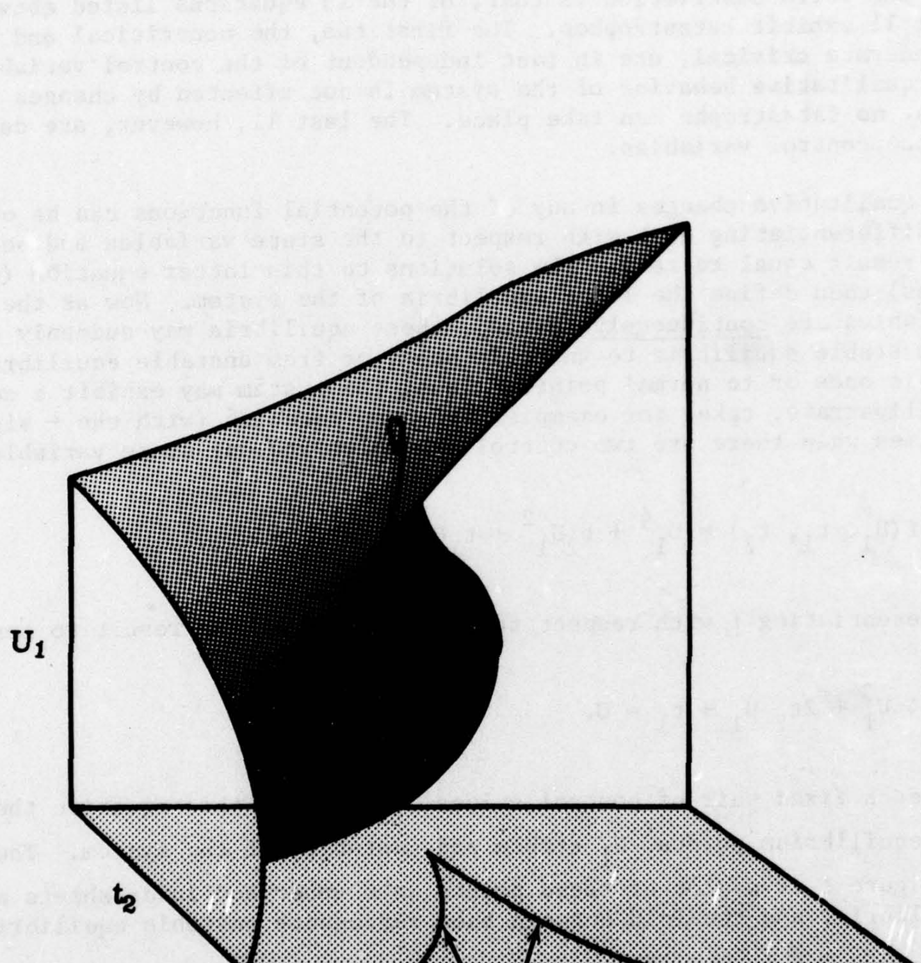


Figure 1. The cusp catastrophe. The system's equilibria (U_1) are shown as a function of the two control variables (t_1,t_2).

The edges of the cusp in the control space define the bifurcation set, the points that satisfy the equation $4t_2^3 + 27t_1^2 = 0$. Moving across this bifurcation or catastrophe set produces or annihilates one of the two possible stable equilibria in the system. If the system is resting at an equilibrium that is then annihilated, the system moves discontinuously toward the surviving equilibrium, and a catastrophe is realized.

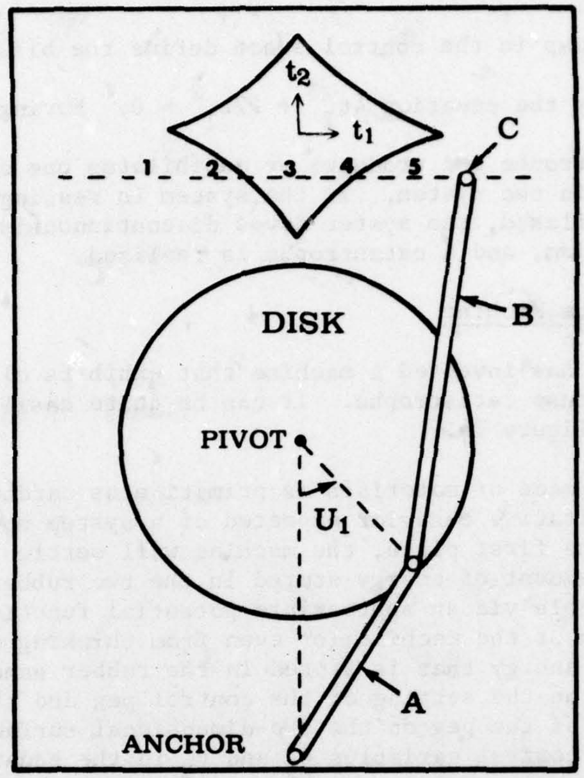
Zeeman's Catastrophe Machine

Zeeman (1976a) has invented a machine that exhibits clearly all the properties of the cusp catastrophe. It can be quite easily constructed, as can be seen in Figure 2a.

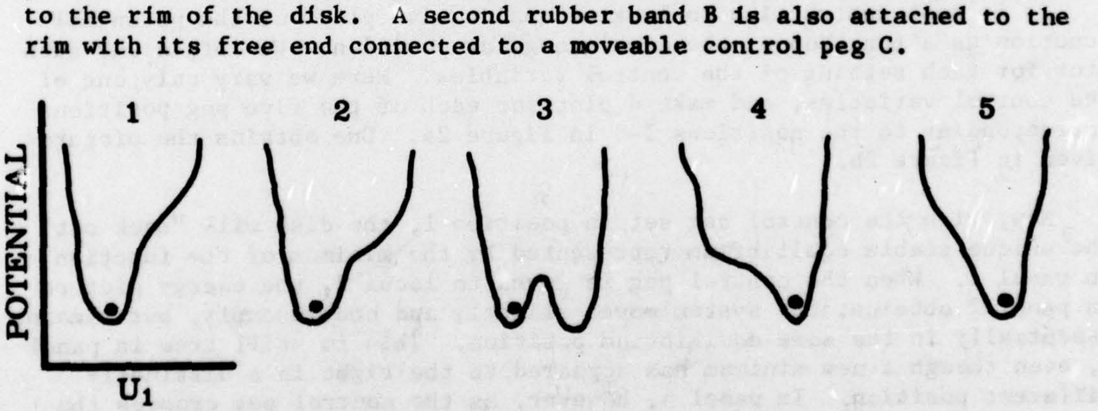
Even a machine made of materials as primitive as cardboard and thumbtacks will show the qualitative behavior expected of a system modeled by the cusp catastrophe. In the first place, the machine will settle in stable states that minimize the amount of energy stored in the two rubber bands; hence, it must be describable via an appropriate potential function. It is clear from an examination of the machine (or even from thinking about the diagram) that the amount of energy that is stored in the rubber bands at any given moment will depend on the setting of the control peg and the position of the disk. The setting of the peg on the two-dimensional surface of the machine corresponds to the control variables t_1 and t_2 in the equation of the cusp catastrophe, and the angular position of the disk corresponds to the state variable U_1.

It is most instructive to look at qualitative plots of the potential function as a function of the disk's angular position. We obtain one such plot for each setting of the control variables. Here we vary only one of the control variables, and make a plot for each of the five peg positions corresponding to the positions 1-5 in Figure 2a. One obtains the picture given in Figure 2b.

Now, with the control peg set in position 1, the disk will "seek out" the unique stable equilibrium represented by the minimum of the function in panel 1. When the control peg is moved to locus 2, the energy picture in panel 2 obtains; the system moves slightly and continuously, but remains essentially in the same equilibrium position. This is still true in panel 3, even though a new minimum has appeared to the right in a distinctly different position. In panel 4, however, as the control peg crosses the bifurcation set a second time, the machine jumps to a new equilibrium, for the old equilibrium has now disappeared. This sudden jump to a new stable position, as a function of a small, continuous change in the value of a control variable, is a realization of the catastrophe. In panel 5, a unique minimum again obtains, this time at the new equilibrium.



a. **The catastrophe machine: A cardboard disk is free to rotate around its pivot. One rubber band A is anchored to the machine's base and attaches to the rim of the disk. A second rubber band B is also attached to the rim with its free end connected to a moveable control peg C.**



b. **Qualitative changes in the potential function $U_1^4 + t_2U_1^2 + t_1U_1$ as produced by changes in the control variable t_1. The vertical position (t_2) of the control peg is fixed as its horizontal position (t_1) is moved through the loci 1-5 in part (a) of the figure. The disk comes to rest at the rotation angle U_1 for which the potential function is a minimum.**

Figure 2. Zeeman's catastrophe machine and forms of its associated potential function.

One can, of course, now reverse the direction of movement of the control peg, and a catastrophe will become manifest at position 2. One may wish to call the alteration in equilibrium conditions that occurs upon first crossing into the cusp a "silent catastrophe." For the system indeed has changed its type or qualitative structure, although no change is yet manifest in the system's behavior. Certainly the changes in the character of the potential function are equally dramatic upon entering versus leaving the cusp. And this is true whether the entrance and exit are on the same side of the cusp or not.

However, in observations of a real system, there is typically no way to ascertain a catastrophe's occurrence in the absence of overt sudden changes of state. These occur only when the stable equilibrium in which the system has been resting now disappears; any number of covert, or silent, catastrophes affecting other minima could occur but would never be observed. But consider a "catastrophe" in the everyday meaning of the word. An airplane crashes. A dam breaks. We see the final system failure as the catastrophe. Yet the objective of post-mortem investigations may be regarded as an attempt to identify the conditions--the design or procedural error--that, in fact, made the ultimate observed "catastrophe" inevitable.

Qualitative Characteristics of the Cusp Catastrophe

Poston and Woodcock (1973) analyzed Zeeman's machine and found that it is indeed a realization of the cusp catastrophe. Zeeman himself suggests a search for other possible realizations and, as a guide, he notes five important qualitative features in the behavior of the machine. These, of course, follow from the nature of the cusp catastrophe surface.

First, the behavior of the system will be bimodal over part of its range. We see this in that folded portion of Figure 1, which graphs the equilibrium states that obtain interior to the bifurcation set. Correspondingly, the machine may be in either of two distinct equilibria such as those that exist in panel 3 of Figure 2b. The actual equilibrium depends on the system's state when the control variables last crossed the bifurcation set.

Second, sudden transitions from one state to another are to be expected. In Figure 1, the state of the system will "fall" down or "jump" up from one surface to another as the values of the control variables move across the cusp from one side to the other. In the machine, the disk jumps from its old position to a new one as the control peg crosses into and then out the other side of the cusp. The suddenness of the change, of course, depends upon how quickly the dynamic influences the system. If a catastrophe machine were constructed with a one-ton disk and two small rubber bands, the speed of movement to a new equilibrium might not be very impressive.

Third, the nature of Figure 1 leads one to expect a hysteresis (delay) effect. That is, the transitions between equilibria should not occur at the same point in the control space when the values of the control variables are changed in different directions. Indeed, the equilibrium "falls" at the left edge of the cusp if the cusp is entered from the right—but "jumps" at the right edge if entry was from the left. However, this should be qualified by noting that hysteresis is not taken as a universal property of the cusp catastrophe. "Maxwell's rule," for example, states that the potential

must always seek the <u>lowest</u> available minimum. The effect of this convention would be to make the "fall" and the "jump" occur at the same point--in the present case in the <u>middle</u> of the cusp.

Fourth, there should be certain state values which are "<u>inaccessible</u>." When one is within the cusp delineated by the bifurcation points, the mid-region of states cannot be stably achieved by any setting of the control variables. The system, as illustrated both by the catastrophe surface of Figure 1 and by the catastrophe machine, quickly leaves the inaccessible region for the stability of the most attractive minimum.

Fifth and last, the system may come to rest in widely <u>divergent</u> final positions, depending on very small differences in initial values of the control variables. Thus, in Figure 1, if one begins at the apex of the bifurcation set and moves around it slightly to the <u>right</u> and then comes forward, the system will come to rest on the upper sheet. However, if one moves around the apex to the <u>left</u>, the system will come to rest on the lower sheet of the surface.

Zeeman (1976a) suggests that, when one or more of the above properties is observed in an empirical system, the possibility that the system might be modeled by a cusp catastrophe should be considered. There is a great deal of disagreement about just how demanding one should be in suggesting the application of catastrophe theory to empirical phenomena. There also seems to be some misunderstanding of just how successes and failures of applications should be interpreted.

<u>Levels of Rigor in Applying Catastrophe Theory Models</u>

Kolata (1977), by entitling a commentary "Catastrophe theory: The emperor has no clothes," seems to imply that shortcomings in applications of catastrophe theory somehow reflect shortcomings in the theory itself. Various critics are quoted as saying that insufficient attention has been paid to scientific details of applications, that definitions are ambiguous or inconsistent, and that alleged proofs in these applications are not proofs at all. However, even if these criticisms are correct--and we believe that they generally are--they have nothing whatever to do with the correctness of catastrophe theory. As a close analogy, an indictment of a particular use of differential equations should not alter one's belief in the correctness of the mathematics of differential equations.

Steen (1977) says "The mathematics of catastrophe theory is beyond reproach." No more should be asked of it. Even if current applications are not promising, it is always possible that future applications will be more successful. To illustrate, Boolean algebra was developed in mid-19th century, but it was not extensively applied until the advent of computers in the mid-20th century.

In the following discussion, therefore, we merely wish to present a scheme for evaluating levels of rigor in present and future applications. We do believe that catastrophe theory will eventually be a useful tool for application in the behavioral sciences, and presumably there is a virtual continuum of levels of rigor in such applications. However, we think it is useful to distinguish roughly among five levels of rigor.

First, one may use catastrophe theory as a loose metaphor only. We may place in this category the statement by Thom (1972/1975, p. 325): "Is not day-dream the virtual catastrophe in which knowledge is initiated?" In such appeals, there is no attempt to show what sort of catastrophe might be involved, nor is there any discussion of the relationship of either dream or knowledge to the mathematics of catastrophe theory. Control variable(s) and state variable(s) are not mentioned, let alone identified.

Second, with increased rigor, it may be demonstrated that the phenomena in question manifest one or more but not all of the qualitative properties of the applicable catastrophe model. In the case of the cusp catastrophe, this would mean that the phenomena exhibit either bimodality, sudden transitions, divergence, inaccessibility, or hysteresis. Each more complex catastrophe would have some analogous set of properties.

Third, the phenomena may be shown to possess <u>all</u> the <u>qualitative</u> characteristics of the type of catastrophe surface being used to model them.

The fourth level of rigor demands the third and, in addition, that at least some properties be quantitative. This is a less stringent demand than might be supposed. For catastrophe theory does not demand that the data <u>as collected</u> be fit by the equations describing the catastrophe surface; rather, it only stipulates that there be <u>some transformation</u> of the axes that results in (for example) the cusp catastrophe surface. Although this allowance for transformation lessens the magnitude of the shift from qualitative to quantitative, there is, to be sure, no guarantee that the investigator can find appropriate transformations.

The fifth level of rigor would, of course, demand a completely quantitative fit between a particular type of mathematical catastrophe and a particular set of empirical phenomena. Unfortunately, the problem of fit at all levels above the lowest is complicated by the fallibility of the data. "Fit" becomes a probabilistic rather than a deterministic problem. In the so-called "softer" behavioral sciences, this problem may be particularly acute because of the likelihood of high variability in the data of observation and even of experiment. Despite such difficulties, Lewis (1977) reports: "I am currently investigating several correlation and regression procedures that can indicate the existence and location of a cusp or butterfly catastrophe." These and other statistical procedures that might permit comparing goodness of fit for catastrophe models versus other models are necessary if one is to determine whether or not catastrophe theory provides the "best" models for particular applications.

APPLICATIONS OF CATASTROPHE THEORY

Review of Previous Attempts to Apply Catastrophe Theory

Zeeman (1976a) has put forth the greatest number of possible applications of catastrophe theory to the behavioral sciences. We first sketch each of the behavioral applications he suggests in this most accessible article, and comment on the apparent level of rigor of each.

1. Aggression in dogs. Zeeman suggests that the fight-or-flight behavior of the dog (to be taken as the state variable) can be described with a cusp catastrophe model in which fear and rage are taken as the control variables. He argues that the transitions from retreating to attack, and vice versa, are sudden, and that the behaviors, in fact, exhibit all the characteristics of a cusp catastrophe. Fear is to be indexed by the degree to which the dog's ears are laid back; and rage, by the degree to which his teeth are exposed. No equations, however, express the relationship between the indicants and the hypothetical variables "fear" and "rage." No data are given to demonstrate relationships between behavior and either the observed or the hypothetical control variables. Thus, no quantitative demonstration has been provided. It is not even clear that flight and attack behaviors lie on a single dimension, or that they take on continuous values. Thus, even the qualitative features of catastrophe theory remain undemonstrated. This application seems to belong to the metaphorical (lowest) level.

2. Fighting in fish. The state variable (flight or fight) is the same as above. Suggested control parameters are the size of the opponent and the distance from the nest (or center of defended territory). The criticisms and conclusion are similar to those above, except that the nature, and therefore scalability, of the control variables is clearer. Nonetheless, any application for which no data are provided must remain metaphorical.

3. Conflict during arguments between people. The state, or behavioral variable is intensity of conflict; anger and fear (as in the dog) are the control variables. No data. Metaphorical.

4. Change in mood. Mood, ranging from self-pity through normal to anger, is taken as the state variable with anxiety and frustration as control variables. It is not clear that any of these variables is measurable, or even a unitary dimension. No data. Metaphorical.

5. Stock market behavior. The state variable is the rate of change of the market index (which can, of course, be a positive or negative number); the control variables are the amount of excess demand and the speculative content of the market. With appropriate definitions, all of these variables should, in principle, be measurable, although it is not at all clear that the two control variables could, in practice, be measured. No data. Metaphorical.

6. Hostilities between nations. A cusp catastrophe is first proposed in which the state variable is "mode of action" and ranges from surrender through full-scale attack. Threat and cost are the proposed control variables. It is not clear that all the values of the "state variable" lie on the same dimension, and threat and cost might well be impossible to measure. No data. Metaphorical.

Zeeman then suggests that a butterfly catastrophe may better describe this situation. Two additional control variables are introduced, but he identifies these only as a bias factor and a butterfly factor. Zeeman says, "The effect of the butterfly factor is to create the third stable mode of behavior" (p. 80). This statement is somewhat disturbing, since mathematical models cannot literally create behaviors (except perhaps in mathematicians). This suggested application is only an unclear metaphor, and increases our skepticism regarding the relevance of the simpler cusp catastrophe to the topic of hostilities between nations.

7. Anorexia nervosa. A behavioral variable ranging from fasting to gorging is taken as the state variable; hunger, "abnormality" of the anorexic's attitude toward food, loss of self-control, and a bias factor are to be the control variables. The state variable seems easily measurable, but again the measurement of the control variables poses difficulties. A special problem could be a possible nonindependence of "loss of control" from the state variable. No data. Metaphorical.

In another paper, Zeeman (1976b) models, with a cusp catastrophe, the misestimates of speed caused by alcohol. The two control variables are "actual speed" (in a driving simulator) and "selectivity" (a hypothetical tendency to pay attention to only part of the cues available for determining speed). The state variable was "estimated speed." The actual speed could be obtained directly from the simulator. The index of selectivity was taken to be the score on the Bernreuter scale of introversion-extraversion, with introverts taken to be more selective. There was no measure of estimated speed in the experiment.

This is a more serious effort to apply catastrophe theory than the attempts previously cited, but it is quite unconvincing due to the lack of any direct measure of the state variable. Zeeman proposes several ingenious but implausible hypotheses to connect his observed (and hypothetical, or transformed) control variables and his hypothetical state variable. Nonetheless, there is a "reasonable" fit between his postdicted theoretical plot and the observed data. The conservative view must be that this example, too, is analogical; but, if one were to accept all of Zeeman's assumptions, it would become an example of a quasi-quantitative fit (level four above).

Zeeman, Hall, Harrison, Marriage, and Shapland (1976) offer still another application of the cusp catastrophe, here as a model for prison riots. Tension and alienation serve as the control variables, and the state variable is taken to be the occurrence of a riot, or some milder protest. All the variables are measurable, but none of the measures is very appealing, and there are only a few observed values of the state variable. In order to make the data fit the model reasonably well, two quite different locations of the cusp were used at different points in time. The overall effort was, of course, post hoc, and is not very convincing. However, the attempt to measure all variables was certainly a step in the right direction. A further step would be the determination of all measures in advance, thus eliminating any possibility of selective bias.

Frey and Sears (1978) introduced a response rule based on catastrophe theory as part of a model for conditioning, and they presented some experimental data that appear to exhibit bimodality, as required by the cusp catastrophe. Most of the numerical data presented, however, are derived from computer simulation rather than from experiment. The manifestation of the required catastrophic properties is logically guaranteed by the arbitrary use of the catastrophe rule for response mapping.

Frey and Sears present no rationale for adopting the response rule from catastrophe theory, but Hillix and Hershman (Note 1) have suggested a rationale based on Amari's (1972) studies of the stability properties of selected logical nerve nets. If the predictions from Frey and Sears' model fit additional experimental data, it will constitute the most convincing application to date of catastrophe theory in the behavioral sciences.[1]

The above review has pointed out various difficulties in extending catastrophe theory to behavioral phenomena. In the physical realm, we have already seen, however, that Zeeman's machine is a rigorous realization of the cusp catastrophe. The buckling of bridge beams and the formation of light caustics can also be modeled precisely. We next discuss the several conditions that must be met, if similarly valid applications are to be obtained in the behavioral realm.

Requirements for Modeling Behavioral Phenomena with Catastrophe Theory

Since any behavioral system will likely manifest a large number of properties, it is essential to choose the "right" properties (or their "correct" combination) as the state variable(s). Similarly, a large number of manipulations may potentially affect the qualitative behavior of the system. An appropriate set of these--precisely two in the case of the cusp catastrophe--must be selected as control variables. Thus, if other variables affect the conditions of stability of the system, they must be held fixed. That is, they must be treated as part of the background of "system" variables not directly examined in the study. There is no ready guide to such selection. Rather, the dual viewpoints of the system scientist and the behavioral researcher must meld (together, no doubt, with substantial trial and error) to accomplish a satisfactory or even promising line of attack.

All of the variables under study should be measured on an interval scale, such that the mathematics of real numbers can be reasonably applied. Ideally, the control variables should be <u>manipulable</u> so that a genuine experiment can be performed.

One must then settle upon a suitable criterion for deciding that the system has reached an equilibrium state. The behavior of individuals, societies, or markets often resists stability, sometimes because the measuring techniques themselves involve substantial random variation. Further,

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[1]Although there are some errors in the way that Frey and Sears conceptualize catastrophe theory and its relation to experimental data, these do not affect the general conclusions.

the system dynamic may sometimes be quite slow, so that even approximate stability is never seen within a reasonable time after changes in control variables have been introduced. We will see an example of such a "slow" experimental system later.

One must also be able to vary the values of the control variables systematically and demonstrate that the stable system states are describable by a catastrophe surface. Since catastrophe theory predicts bimodality for some values, these variations should be not only systematic but directional, rather than a random selection of values of control variables. Note that if values of the control variables interior to the bifurcation set were presented at random, we would expect a bimodal distribution of stable states. Given the typical random variation of behavioral outcomes, however, this bimodality could remain undetected and be regarded simply as error variance.

Finally and ideally, one should show that a model derived from catastrophe theory fits better than the best alternative model. It may, in practice, be impossible to demonstrate this for any given case; in particular, it could be that some quite different underlying theory would predict that the stable states of the system should be fitted by the same equation as the one derived from catastrophe theory.

New Attempts to Apply Catastrophe Theory

An Observing Response Experiment

Keeping in mind some of the above desiderata for demonstrating the applicability of catastrophe theory, we set out to design an appropriate behavioral experiment. Of course, we sought an application in which a discontinuous change in behavior would be likely. It has been repeatedly noted in vigilance experiments (Buckner & McGrath, 1963; Davies & Tune, 1969) that vigilance performance declines dramatically over time. We hoped that a fine-grained analysis would reveal that the decline was discontinuous when plotted as a function of the appropriate (unknown) control variables.

Rational analysis indicates that the monitoring behavior of a mathematically ideal observer ought to be a function of (1) the cost of each observation, (2) the "stakes" involved in subsequent queries about the situation being monitored, and (3) the expected frequency of such queries.

The approach would be conceptually simple: (1) choose two of the above as control variables, thereby making the third into a system variable, (2) set the two control variables at extreme values and administer the monitoring task until the subject's observing behavior stabilized, (3) move to a new pair of values of the control variables until behavior again stabilized, (4) continue until the control space has been systematically and directionally sampled, and (5) examine the fit of a cusp catastrophe to the observed data.

We arbitrarily treated the expected frequency of querying, or "probing" the subject as a system variable. That left both the cost to the subject of making an observation and the stakes as control variables. The choice of a state variable was more difficult. We knew that we wanted to measure <u>some</u>

aspect of the observing response, but not <u>which</u>. This was eventually dictated by the actual monitoring task, which we now describe.

The task was controlled by NAVPERSRANDCEN's PDP-12 laboratory computer; a Tektronix 4006-1 graphics display terminal served as the subject station. Apart from decisions made by the subject, each monitoring trial lasted 60 seconds. The passage of time was marked by a sequence of dots (one per second) on the computer-driven display.

The signal to be monitored was simply ON or OFF; it could change state from one second to the next, but not within a second. The signal was ON in the first second with probability = 0.5. Its subsequent states were then determined by the probabilities in the following transition matrix, where N refers to the current second.

		STATE (N + 1)	
		ON	OFF
STATE (N)	ON	.8	.2
	OFF	.2	.8

The subject could observe the process (learn if the signal was now ON or OFF) by pressing a key at the display station. Each such observation cost a fixed number of "points." At random times, the subject was queried as to the signal's current state; observing in the midst of such a probe was, of course, prohibited. Symmetric stakes always applied so that, for example, the subject might earn 50 points for a correct decision or lose 50 points for an incorrect decision. Resources spent in observing were subtracted from winnings (if any), and the net returns were displayed to the subject at the end of the trial. The three relevant variables--cost, stakes, and expected frequency of queries--were also displayed at the outset of each trial.

How should an ideal observer (who is to maximize expected returns) behave in this complex task? After considerable analysis, we discovered that the optimal strategy is to observe cyclically (if at all); that is, to "refresh" the information about the signal every $\underline{k}$ seconds. Of course, $\underline{k}$ depends jointly on the three cited variables--being inversely related to the stakes and the probability of a query, and directly related to the cost of looking. Further, when probed, the ideal observer always reports that the signal's current state is its last known state. This result follows from the transition matrix above. These prescriptions had little direct relation to catastrophe theory; however, they suggested that we should measure the moments of the distribution of interobservation times and use the "best" of these moments as the state variable.

Two fatal flaws appeared when the experiment was actually run. First, we did not get stable behavior, except when it was just no behavior (i.e., no observing). For one thing, we had chosen a situation with a "slow dynamic." We had sought to examine the effects of systematic sequences of control variables on equilibrium behaviors, but we could not obtain even one stable behavior (ignoring the no-observing case) within the limits of an experimental session. Thus, the experiment was terminated. The lesson was clear: If catastrophe theory is to be tested by experiment, one must choose a situation with a "fast dynamic"; that is, one in which stable states are quickly achieved.

Our experiment also had a second weakness that we did not foresee. It was that "chance" substantially determined the outcome of a particular trial for the subject. That is, the subject might make lucky (or unlucky) guesses about the state of the signal, and win or lose unexpectedly large numbers of points. In more formal terms, we had not taken sufficient care to have stable "attractors." When a subject was "lucky," there was nothing to repel his behavior from that point; conversely, if his behavior were nearly optimal but he were unlucky, then his behavior might be repelled from the very point to which we would have it attracted. Thus, long-term behavior should have been forced toward the ideal, but the inherent random trial-to-trial variations radically perturbed the process. The lesson was again clear: Large random variations in the "dynamic" can preclude the emergence of stable behaviors. To put it another way, perhaps we did not give the subjects an accurate reading of the values of the "potential function" to be minimized. Such shortcomings must be eliminated.

As a final aside, we note that most subjects did not observe cyclically or even approach such a schedule. Also, most observers, when probed, did not always respond with the last known state of the signal (as the optimum tactic demands). Rather, they often tried to "outguess" the odds by guessing that the signal had changed state since it was last observed.

Attempts to Control Reversible Depth

In our second attempt to apply catastrophe theory, we tried to correct the above errors. We still needed behaviors which seemed to change discontinuously, and we now knew that we needed phenomena which settled quickly into a stable condition. Now, the time scale (i.e., dynamic) of perceptual phenomena is clearly much faster than that for complex learning, and there are many percepts that appear to undergo rapid changes. Price (1969) and Taylor and Aldridge (1974) have published recent articles on reversible perspective, a phenomenon in which such changes in depth "spontaneously" occur in the presence of a constant stimulus. Since one perspective quickly appears when the stimulus is presented (the dynamic is fast), and another appears suddenly when the first disappears (a possibly catastrophic change), we attempted to control these percepts and see if their changes occurred in accordance with the cusp catastrophe model.

An example of reversible perspective is the familiar instance of the "moon craters"; sometimes the craters appear as they "should," as indentations, and at other times they appear as bumps or elevations. However, there are no absolute cues on a photograph that ensure that either perception is "correct."

Thus, in Figure 3, if the illumination were coming from above, craters would have produced the observed pattern of light and shadow; if the illumination were from below (turn the figure upside down), however, bumps would have produced the same observed pattern.

A fair conclusion is that if the perceived position of the illumination can be controlled, then the perspective (and thus the percept) should be controllable. From this point of view, the usual spontaneous reversals in perspective would be regarded as random changes in the (unconscious) "decision" as to the light's present locus (for actual objects) or past locus (for photographs).

Genuine craters were to be photographed, and the physical orientation (angular rotation) of a selected crater was taken as the first control variable. It seems logical that the photographed crater would be perceived correctly when its orientation matched that of the original scene, and would be perceived as a bump, or hill, when its orientation was reversed (180° away from its original direction). Further, we hoped that as the crater's orientation was changed gradually, the reversal of perspective would manifest the same hysteresis effects predicted from the cusp catastrophe. It was already clear that intermediate perspectives were unavailable, that sudden changes occurred, and that perception was bimodal--three of the qualitative properties of a catastrophe. These three properties, incidentally, seem to be shared by all reversible figures.

If we succeeded in controlling perspective by manipulating the orientation of the critical figure with respect to the light source, we hoped to use actual depth (of a crater) or height (of a hill) as a second control variable; on a photograph, this depth or height would be represented by different gradients of light and shadow. Perceived depth or height, to be indicated on some measuring device, would be the state variable.

We first constructed a three-dimensional figure from children's modeling clay, in which indentations were made with a large marble. The initial concern was whether such a figure would reverse in perspective. When viewed patiently through a tube with one eye, and under largely directional illumination, it does reverse dramatically. Indeed, when the tube is moved to reveal the plate on which the model rests, even the plate seems to have turned upside down--as should be expected if one's perceptual system has, at the moment of perspective reversal, "decided" that the light source is on the opposite side from its true source.

Given this encouraging outcome--the production of a reversible figure with such simple apparatus--we proceeded to photograph it. Then a part of the figure was cut out smoothly with a shim punch, so that it could be rotated at will within the context of the total picture. The larger portion of the picture was expected to provide the dominant impressions as to the direction of the light when the photograph was taken.

Preliminary results were extremely encouraging. The photograph tended to be "correctly" perceived (as craters) when first viewed, particularly if the source of viewing illumination approximated that of the source of photographic illumination. When a "critical crater" (the excised part of the figure) was rotated within the context of the total picture, it tended to reverse and be seen

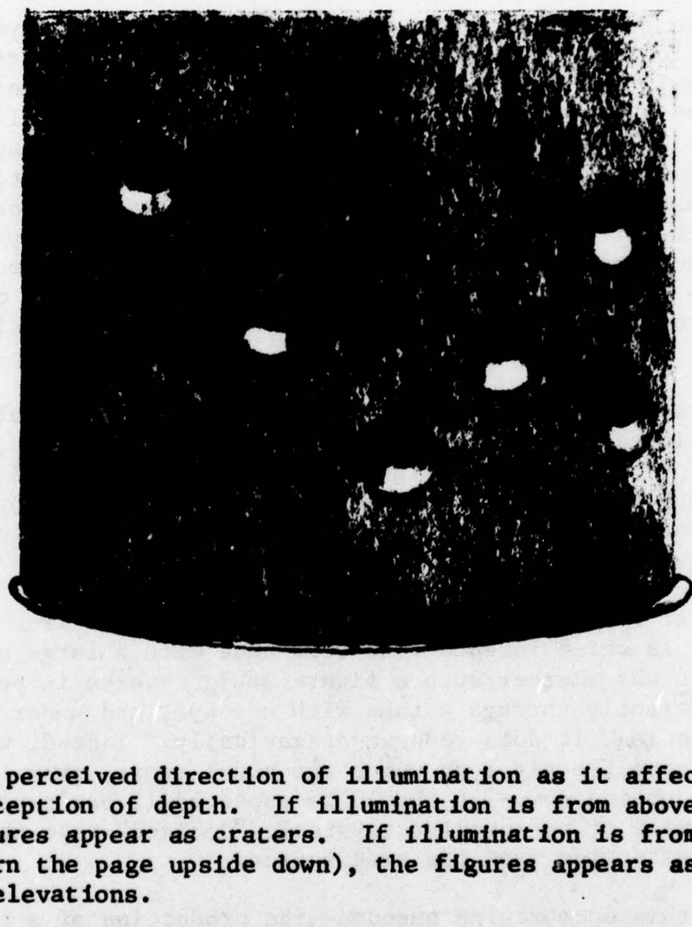

Figure 3. The perceived direction of illumination as it affects the perception of depth. If illumination is from above, the figures appear as craters. If illumination is from below (turn the page upside down), the figures appears as bumps or elevations.

as a "bump" at approximately 90° of rotation; it also reversed back to a "crater" when returned toward its original position. At this point, it seemed that we needed only to add variable depth as a second control variable, and to run experiments to test for the predicted hysteresis and divergence effects.

Unfortunately, difficulties quickly emerged. Most critically, the perception of the background part of the photograph was insufficiently stable. After a period of viewing, even with both eyes open, and with obvious cues revealing the direction of illumination, the background would reverse. If the background refuses to remain stable, it is, of course, meaningless to speak of manipulating the orientation of the critical figure with respect to the perceived direction of illumination.

Since most studies of reversible perspective have deliberately minimized cues for the direction of illumination, we first expected that it would be a simple matter to add background information to stabilize the perception. Accordingly, familiar objects like keys and matches were placed in the scene so that they cast shadows and gave obvious directional cues. Reversals still occurred. Different sorts of craters (holes drilled in wood) were used. Reversals occurred. Map tacks were placed in and around every crater. Reversals occurred. Color photography was substituted for black and white. No help. As a last desperate measure, a marble was placed in one crater. The whole figure (except for the marble) still reversed, and even the crater holding the marble became a hill--with a crater just large enough to hold the marble!

A simple viewing apparatus was then constructed in a further attempt to control the perceived direction of illumination. A single lamp illuminated the photograph from above, and the photograph could be viewed either binocularly or monocularly. Nothing helped. Again, unintended and uncontrolled reversals occurred.

The situation might still have been salvageable if, when the perception of the background switched, the perception of the critical figure also switched in a consistent way. However, it appeared possible for observers to hold inconsistent perceptions; after some practice in observing the rotatable figures, they might report "all craters," even when the critical figure was rotated 180° from its original position. At this point, it seemed hopeless to continue working on the determination of the "stable states" of this kind of perception. It remains a possibility that stability could be achieved with actual three-dimensional figures, binocularly viewed. Concealed sources of lighting for a part of the figure might then provide the control variable. Such deception might serve to control perception, but we did not build the sophisticated apparatus required to test this speculation.

Another related, and untested, possibility is that techniques of the type developed by Julesz (1971) could be modified so as to produce reversible perspective with better stability. The potential state variable in this case would remain perceived depth, and the control variables would be (1) the amount of offset (one direction producing depth, the other height), and (2) the proportion of elements shifted in each direction.

The Reversible Movement Experiment

We next encountered another reversible perception that "promised" to solve our problems with stability. Attneave (1971) stated that a pattern of four lights can be arranged to achieve reversible apparent motion. Thus, with constant stimulation, sometimes vertical movement, and at other times horizontal movement, would be reported. The necessary arrangement is illustrated in Figure 4.

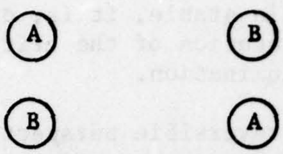

Figure 4. A simple arrangement of four lights that produces reversible apparent movement. The lights are identical. The pairs A and B are alternately presented for 100 msecs each, with 100 msec blank time following the display of each pair.

All four lights are identical, but if the pair denoted A is presented so as to form the ends of an imaginary diagonal line, and the pair denoted B so as to form the opposite diagonal, apparent movement should be seen (provided, of course, that the time intervals are properly chosen). The direction of perceived movement should be easily manipulable via two control variables: (1) spacing in the horizontal (X) dimension, and (2) spacing in the vertical (Y) dimension. Thus, if the X spacing were great and the Y spacing small, then vertical movement should be seen, and conversely.

It was straightforward to arrange the display on a CRT controlled by the Center's PDP-12 laboratory computer. Although the decay times of the phosphor were not ideal, the display, when viewed under ordinary room illumination, was quite adequate for our purposes. The basic display sequence was as follows: the pair A was illuminated for 100 msec followed by a 100 msec blank period. Then the pair B was illuminated for 100 msec followed by a 100 msec blank period. This sequence was then presented repeatedly. The resulting series always produced good horizontal or vertical movement.

We took as the idealized state variable the probability that vertical movement was seen. In fact, the observed proportion of "vertical" reports was the actual datum. It was originally intended to take as the two control variables the X and Y separations that corresponded to the positional control variables in Zeeman's catastrophe machine. However, we were quickly convinced that the simple ratio of X to Y separation, rather than the individual separations, was the controlling factor in the perception of a given movement.

It is clear that all objectively "square" configurations have $X/Y = 1.0$, and, neglecting minor bias factors, such stimuli presumably present the maximum opportunity for reversible movement. All cases with $X > Y$ should have a greater tendency to produce vertical than horizontal movement; and vice versa for stimuli with $Y > X$. Thus, in this context, we regarded the movement-perception system as having a single control variable, X/Y. We assumed that the unidentified second control variable could be treated as a constant.

We proceeded to investigate the nature of the sudden transitions that we were sure could be produced by presenting stimuli that varied in their X:Y ratio. The general procedure was simple: Present a stimulus and ask a subject to press one key if vertical movement was perceived, and another if horizontal movement was perceived. A new stimulus was presented as soon as the subject made a response.

Sequential displays. In the first set of observations, systematic sequences of stimuli were presented, beginning with one of the extreme X:Y separation ratios, and proceeding in a series of 31 steps until the opposite extreme was reached. In particular, stimulus #1, which had an X:Y ratio of 1:6, was the "most horizontal" display and was the starting point for the so-called *H* sequences. Stimulus #31 (ratio of 6:1) was the "most vertical" display and was the starting point for the so-called *V* sequences. Ten sequences of each type were presented, with five repetitions of the series *H V V H* comprising a single experimental run. Stimulus #16 was the square.

We certainly hoped to find hysteresis--with the transitions from "horizontal" to "vertical" reports in the *H* sequences occurring at higher stimulus numbers than the transitions from "vertical" to "horizontal" in the *V* sequences.

Twelve subjects viewed the sequential displays binocularly. The mean stimulus numbers for their transitions between directions of apparent movement appear in Table 1. Positive differences between the *H* and *V* sequences suggest a hysteresis effect. It can be seen that eight of the subjects showed some tendency toward hysteresis. The other four observers tended to switch perceptions before the square stimulus was reached, and they did so for both kinds of directional sequences. These latter data were clearly contrary to the hysteresis expectation. Once a subject shifted from one perception to the other, there was, in general, no shifting back. Overall, the small trend toward hysteresis in these binocular-viewing subjects was not significant. The mean difference between the *H* and *V* sequences was +2.23; t (11) = 1.62.

Since Julesz (1971) had reported that monocular and binocular movement receptors exist at different locations in the visual system, we were hopeful that monocular viewing might better reveal a hysteresis effect. Accordingly, four additional observers were run with the non-dominant eye blindfolded.

Figure 5 shows the data for this monocular viewing of the sequential displays. Note that, for both the *H* and *V* sequences, the ordinate was in fact 0.0 for all abscissa values less than those plotted and was 1.0 for all abscissa values greater than those plotted.

For *H* sequences, the displays corresponding to 50 percent vertical reports were stimulus numbers 11.5, 12.5, 13.0, and 14.5 for the four observers A, B, C, D of Figure 5. Thus, the transitions to "vertical" preceded the square display. For *V* sequences, 50 percent vertical responding occurred at stimulus numbers 16.5, 18.5, 18.5, and 15.5 for the same respective observers. Thus, none of these subjects exhibited hysteresis. For observer D, the bidirectional transitions occurred at virtually the same stimulus, and the other three subjects showed clear evidence of "anti-hysteresis." That is, they made anticipatory rather than delayed transitions to the opposite perception.

Table 1

Mean Stimulus Number for the Transition Between Reported Directions of Apparent Movement: Sequential Displays with Binocular Viewing

	Type of Sequence		
Subject #	H	V	H-V
1	13.9	14.7	-0.8
2	18.5	9.4	+9.1
3	13.9	12.5	+1.4
4	13.2	12.6	+0.6
5	13.6	15.8	-2.2
6	17.8	13.4	+4.4
7	17.6	8.3	+9.3
8	12.9	18.2	-5.3
9	13.2	16.7	-3.5
10	18.1	11.9	+6.2
11	14.9	13.0	+1.9
12	17.2	11.6	+5.6

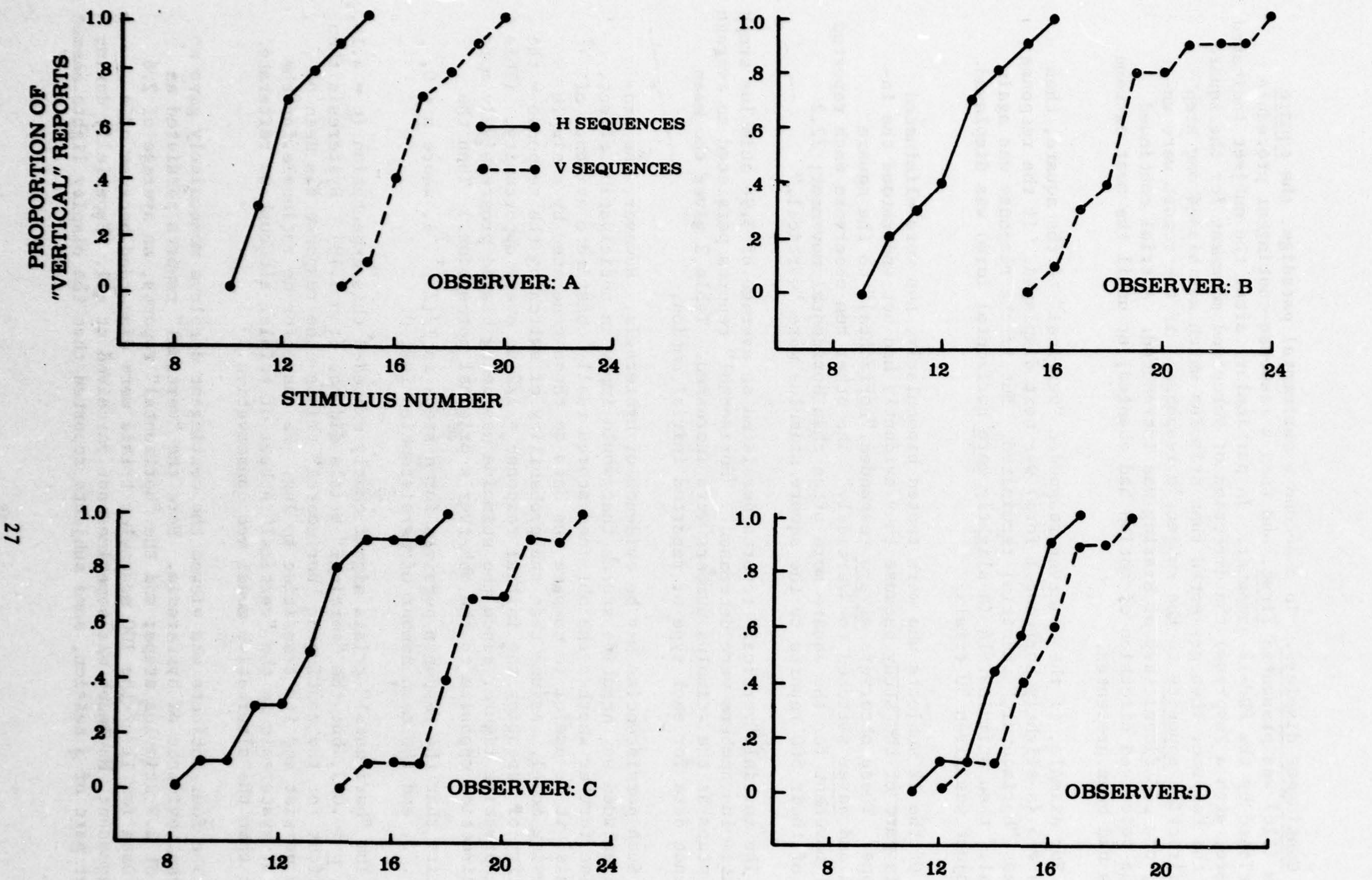

Figure 5. Perceived apparent movement for the sequential displays with monocular viewing. The H sequences started with stimulus #1, the "most horizontal" display; V sequences started with stimulus #31, the "most vertical" display. For both H and V sequences, the ordinate is 0.0 for all abscissa values less than those plotted and is 1.0 for all abscissa values greater than those plotted.

Contingent displays. In a second experimental paradigm, the square stimulus (#16) was presented first, and then a response-contingent procedure was initiated by the PDP-12 computer. In particular, after the subject indicated (as before, with a keypress) the direction of perceived movement for the square figure, the computer then presented that stimulus which was biased one step in the direction opposite to the subject's response. If the report were unchanged, one additional step of biasing was introduced. A trial continued until the perceived direction of motion had changed, or until the most extreme stimulus had been presented.

For example, if the observer responded "vertical" to the square, then stimulus #15 (a slightly horizontal form) was next displayed. If the response to #15 was "horizontal," the trial terminated. But if the response was again "vertical," then stimulus #14 (a slightly more horizontal form) was displayed. Each subject was given 50 trials.

Of the 12 subjects who were tested binocularly, two were eliminated from this part of the study because they evidently had not understood the instructions. These observers always responded "horizontal" to the square stimulus and never switched to "vertical." The other ten observers each reported vertical movement for the square more often than horizontal movement; 72.2 percent of their 500 reports to the square stimulus were "vertical."

The initial "vertical" reports persisted an average of 3.96 stimulus steps as the stimulus numbers were decreased. "Horizontal" reports persisted an average of 5.31 steps as the stimulus numbers were increased. Table 2 gives the mean persistence data for each type of reported initial motion.

Such persistencies may be evidence of hysteresis. However, the procedure precluded any negative scores that would imply an anticipation effect. To examine further whether the obtained scores really constitute evidence of hysteresis, it is useful to compare the data to those generated by a simple coin-tossing model. Assume that the probability of switching the response = the probability of repeating the initial response = 1/2 at every opportunity. (This is a conservative figure, since the stimulus was being biased progressively more in the direction opposite to the observer's original perception.) Then the probability that the response persists for $\underline{n}$ steps is $1/(2^{n+1})$, where $\underline{n} = 0, 1, 2, \ldots$, and the mean number of persistencies is 1.0.

The "horizontal" trials significantly exceeded this expectation ($\underline{t} = 4.18$, $\underline{df} = 11$, $\underline{p} < .01$), but the "vertical" trials did not ($\underline{t} = 1.18$). Hysteresis then was manifest for the contingent "horizontal" trials--the response was seen to strongly persist and its transition to lag. We can offer no rationale for the absence of hysteresis in the "vertical" subset of trials, although we reiterate the fact that the alternative model was conservative.

The four subjects who viewed the contingent displays monocularly gave no significant evidence of hysteresis. Here the "vertical" reports persisted an average of 2.9 stimulus steps; and the "horizontal" reports, an average of 2.6 steps. Data for 11 of the 100 monocular trials were discarded because of a new phenomenon: Movement was sometimes not perceived at all. Especially during the later part of a session, some subjects reported that the display lights seemed

Table 2

Persistence of the Initial Perception of the Square Stimulus: Contingent Displays with Binocular Viewing

	Reported Movement for the Square Stimulus			
Subject #	"V"	(N)	"H"	(N)
1	3.9	(46)	8.3	(4)
3	3.3	(39)	1.7	(11)
4	4.0	(30)	3.8	(20)
5	3.7	(36)	2.8	(14)
7	10.4	(34)	12.1	(16)
8	1.7	(35)	2.1	(15)
9	0.6	(30)	3.6	(20)
10	2.7	(41)	5.8	(9)
11	2.6	(38)	5.8	(12)
12	6.7	(32)	7.1	(18)

Note: Entries are the mean number of times the initial reported movement of the square was repeated. There were N trials with the indicated movement.

to blink alternately but without movement, would change in brightness, or would disappear completely. Evidently a fatigue-like or habituation-like process in the monocular movement detection system is responsible for this phenonemon. Apparent movement was generally restored if the subject briefly looked away from the display or closed the viewing eye.

Inspection of the data for the subjects observing binocularly indicated a possible relationship between the tendencies to exhibit hysteresis in the sequential and the contingent conditions. For the ten subjects in Table 2, the product-moment correlation between their H-V transition scores in the sequential condition and their persistence scores in the contingent condition was 0.76, $p < 0.01$. Thus, the trends toward hysteresis were significantly correlated in the two procedures, although only in the contingent case was the evidence for hysteresis itself statistically significant.

Random displays. A final random condition is reported here for the four subjects who were tested monocularly. A trial consisted of presenting the 31 different stimuli in random order. The observer indicated the perceived direction of movement, and the next stimulus was presented immediately. There was a 10-sec intertrial rest period and a total of 20 trials.

This procedure produced substantially more variability in perceived motion than did either of the subsets of the sequential and the contingent procedures. This increased ambiguity of the (same) displays was manifest in the number of stimuli that gave rise to both "horizontal" and "vertical" reports. These data appear in Table 3 as the width of the bimodal response region for each of the procedures.

Table 3

Width of the Bimodal Response Region by Procedure:
Monocular Viewing

	Sequential		Contingent		Random
	Type of Sequence		Response to Square		
Observer	H	V	"H"	"V"	
A	4	5	5	3	8
B	6	8	3	4	13
C	9	9	3	4	14
D	5	6	10	6	28

These results seem to conform with the expectation from catastrophe theory. For, on a given trial in both the sequential and contingent procedures, the observer's preceding response(s), together with the systematic nature of the stimulus series, "fix" that observer to just one sheet of the response surface.

Thus, except for the sudden reversals and the inevitable small noise, the perceptual reports should have little variability. In the random procedure, however, there is no sequence, and each stimulus is generally "isolated" from its nearest neighbors. Thus, with the very same stimulus, we may "catch" the observer on either the upper or the lower sheet of the surface. Response variability must necessarily increase.

To summarize the observations of this experiment as they relate to catastrophe theory, there seems to be some (but by no means perfect) indication of four of the five qualitative properties stated by Zeeman: sudden changes, unavailability, hysteresis, and bimodality. As we have seen, the evidence for hysteresis was equivocal. There was insufficient precision of control to even attempt a demonstration of the divergence property. In terms of the "levels of rigor" discussed earlier in this report, the present results meet the demands of level two but clearly fail to reach level three, in which all the qualitative properties of the cusp would be manifest. However, elements of level four (namely, some quantitative demonstrations) are present.

It is possible that a swallowtail catastrophe might provide a more adequate model than a cusp catastrophe. In the swallowtail, there are three control variables and one state variable. The three control variables here might be X/Y separation, bias, and habituation. Although the last two variables are not directly observable, the series of observations suggests their presence. It may be that--like X and Y separation--bias and fatigue are reducible to a single control variable, such as "readiness of the vertical motion detector to respond relative to readiness of the horizontal movement detector to respond." Such issues will have to be resolved through further analysis and additional experimentation.

INVESTIGATION OF MATHEMATICAL NEURAL NETS

Reversible Percepts and Mutual Inhibition

Despite the limited success of our manipulations in the laboratory, we believe that the perceptual realm still holds great promise for the exploration of catastrophe theory. Data can be gathered quickly, stable (or quasi-stable) perception occurs "instantaneously," and the instrumentation required for selected studies is not unduly complex.

There are also indications that the neural systems "beneath" perception might be a promising area for study from the point of view of catastrophe theory. The various reversible phenomena all behave as though they included two systems (percepts) in conflict. It is at least plausible that one neural system underlies each of the two possible percepts, and that the two systems inhibit each other. Thus, in our observation of reversible apparent movement, one never sees "good" horizontal <u>and</u> vertical movement simultaneously (although there are sometimes weak and fleeting "compromise" perceptions). In the reversible depth case, one sees craters <u>or</u> hills but not both simultaneously. A host of other reversible figures behave in analogous fashion.

Other lines of evidence (von Bekesy, 1967) indicate the ubiquity of mechanisms of mutual inhibition, or of lateral inhibition, in which the inhibiting "systems" may not be distinct. Hearing, vision, and touch are probably sharpened by virtue of inhibiting mechanisms that emphasize boundaries between areas that are differentially stimulated. The study of mutually inhibitory systems thus seems amply justified by its implications for general sensory problems, as well as by its apparent connection to reversible phenomena.

Zeeman (1976a) suggests that there is also a connection between such neural systems and catastrophe theory. He believes that neural attractors in the brain must underlie his catastrophe models of behavior. Thom (1972/1975) often mentions "conflicts of regimes" in his discussion of the various kinds of catastrophes. Thus, it seems that the study of potentially conflicting neural nets is likely to help us find out whether or not catastrophe theory can become more than a loose metaphor for behavioral science.

Amari (1972) made a mathematical analysis of selected neural nets and reported on a simple formal net that had stability characteristics exactly like those described by a cusp catastrophe. Significantly, he did not cite Thom in his report and was apparently unaware of his work at that time. Thus, his work, which is continuing (Amari, Yoshida, & Kanatani, 1977), constitutes an appealing argument for the possibility of applying catastrophe theory in a setting relevant to behavior. Ideally, one would like to demonstrate that the stability conditions of the nerve nets were, on the one hand, modeled by catastrophe theory and, on the other hand, such that they "described" the perceptual reports obtained in the laboratory.

Stability Conditions of a Hypothetical Nerve Net

Amari (1972) made an appealing case for representing the macroscopic behavior of a self-exciting nerve net with the simple differential equation:

$$\frac{dS}{dt} = W\Phi(S)-H-S.$$

Here $\underline{S}$ is the activity level (the output) of the net, $\underline{H}$ is its threshold, $\underline{W}$ is a positive constant representing its "self-connection weight," and $\Phi(S)$ is the value of the cumulative normal function at $\underline{S}$.

Amari showed that this net is either monostable or bistable depending on the values of the parameters $\underline{W}$ and $\underline{H}$. Thus, consider Figure 6 from Amari (1972, p. 649). Here, if $\underline{W} \leq (2\pi)^{\frac{1}{2}}$, the net is monostable. If $\underline{W} > (2\pi)^{\frac{1}{2}}$, the net is bistable if

$$\left|(H/W) - \tfrac{1}{2}\right| < f(W)$$

and monostable otherwise where

$$f(W) = \Phi\left\{[\log(W^2/2\pi)]^{\frac{1}{2}}\right\} - \frac{1}{W}[\log(W^2/2\pi)]^{\frac{1}{2}} - \frac{1}{2}.$$

This monostable-bistable character of Figure 6 is evidence of the now-familiar cusp catastrophe. Now take $\underline{W} = 6.0$ and plot with Amari the actual equilibrium state(s) $\underline{S}_0$ as a function of $\underline{H}$, as in Figure 7. For $\underline{H}$ between $3.0\pm\simeq 1.1$ there exist three equilibria, just two of which are stable; for $\underline{H}$ outside this range, there is only one equilibrium and it is stable. However, for fixed $\underline{H}$ and $\underline{W}$, this net will simply settle down into a single stable state regardless of its initial state. Thus, as it stands, it cannot model reversible perception.

However, if we let the upper equilibrium in Figure 7 represent, say, vertical movement and the lower represent horizontal, we can produce oscillations (reversals) from one to the other by providing feedback to the threshold $\underline{H}$. Accordingly, let $\underline{H}$ itself have a differential equal to S/D, where $\underline{S}$ is again the current activity level of the net and $\underline{D}$ is a suitable "damping" constant.

Again, take $\underline{W} = 6.0$ and let $\underline{H}$ be small so that only one equilibrium exists at $\underline{S}_0 > 0$. Let $\underline{S}$ be close to $\underline{S}_0$ and decreasing, for illustration. Then the feedback loop will increase $\underline{H}$, which, in turn, decreases $\underline{S}$ in a trajectory near the upper sheet of Figure 7. The system will next enter the bistable region and evolve slowly until $\underline{H}$ exceeds its critical value of approximately 4.1. A single equilibrium will then remain at $\underline{S}_0 < 0$ and the system will then quickly plummet toward that state. But, with $\underline{S}$ now negative, the feedback loop decreases $\underline{H}$ and hence increases $\underline{S}$ smoothly until a sudden jump is produced in the system at $\underline{H} = 1.9$. The net then continues to oscillate with a slow dynamic on the upper and lower sheets (vertical and horizontal percepts?) and with a fast transition (perceptual reversal?) between them.

We have implemented a digital simulation of such a net (one with feedback to the threshold) on the PDP-12 computer. Figure 8 graphs the trajectory of the state variable $\underline{S}$, where $\underline{W} = 6.0$, $\underline{H}$ is initialized at zero, and the initial value of $\underline{S}$ is +3.0. The damping constants for the $\underline{S}$ and $\underline{H}$ differentials were 40 and 20 respectively.

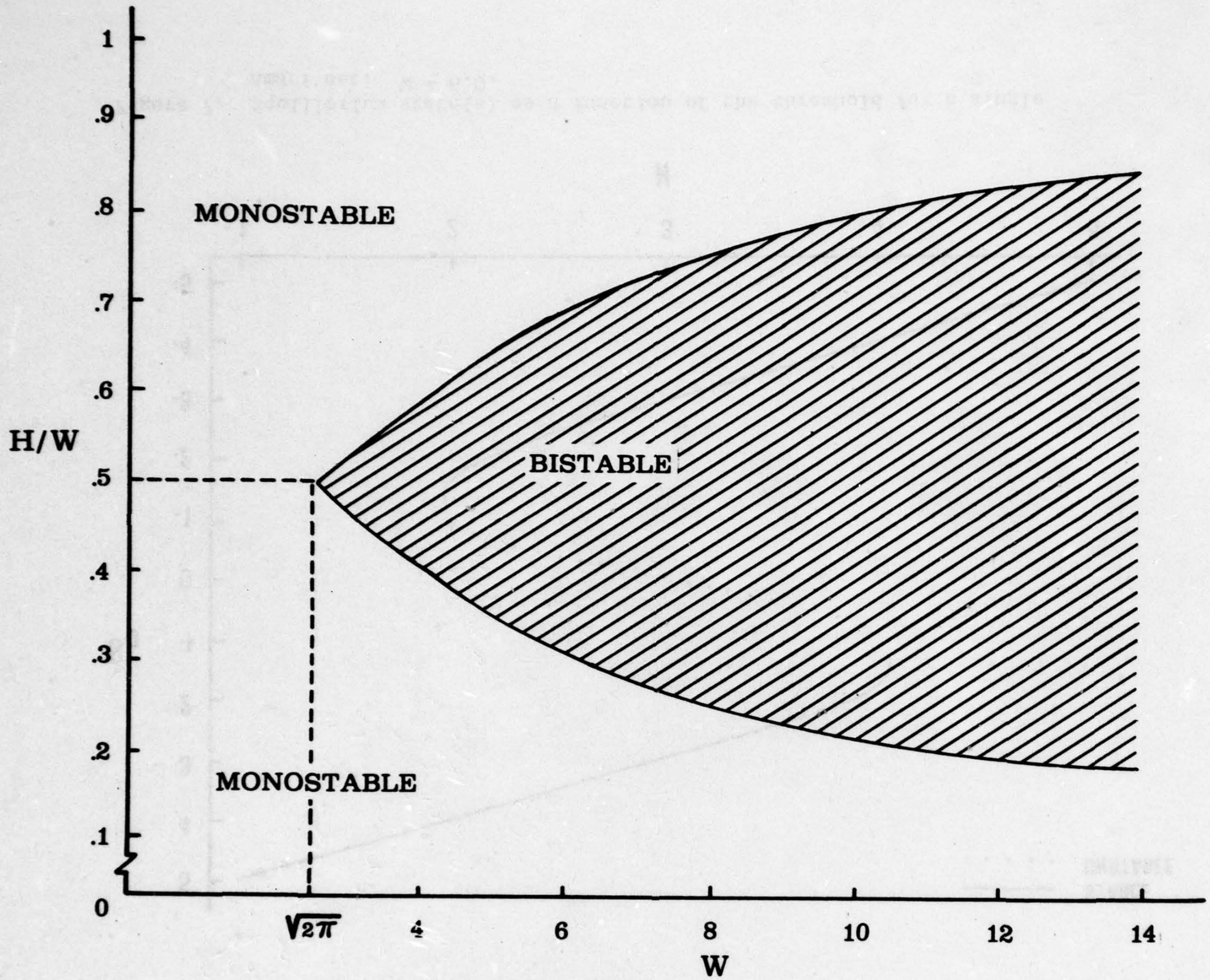

Figure 6. The stability profile of a single Armari net. W is the connection weight and H the threshold of the net.

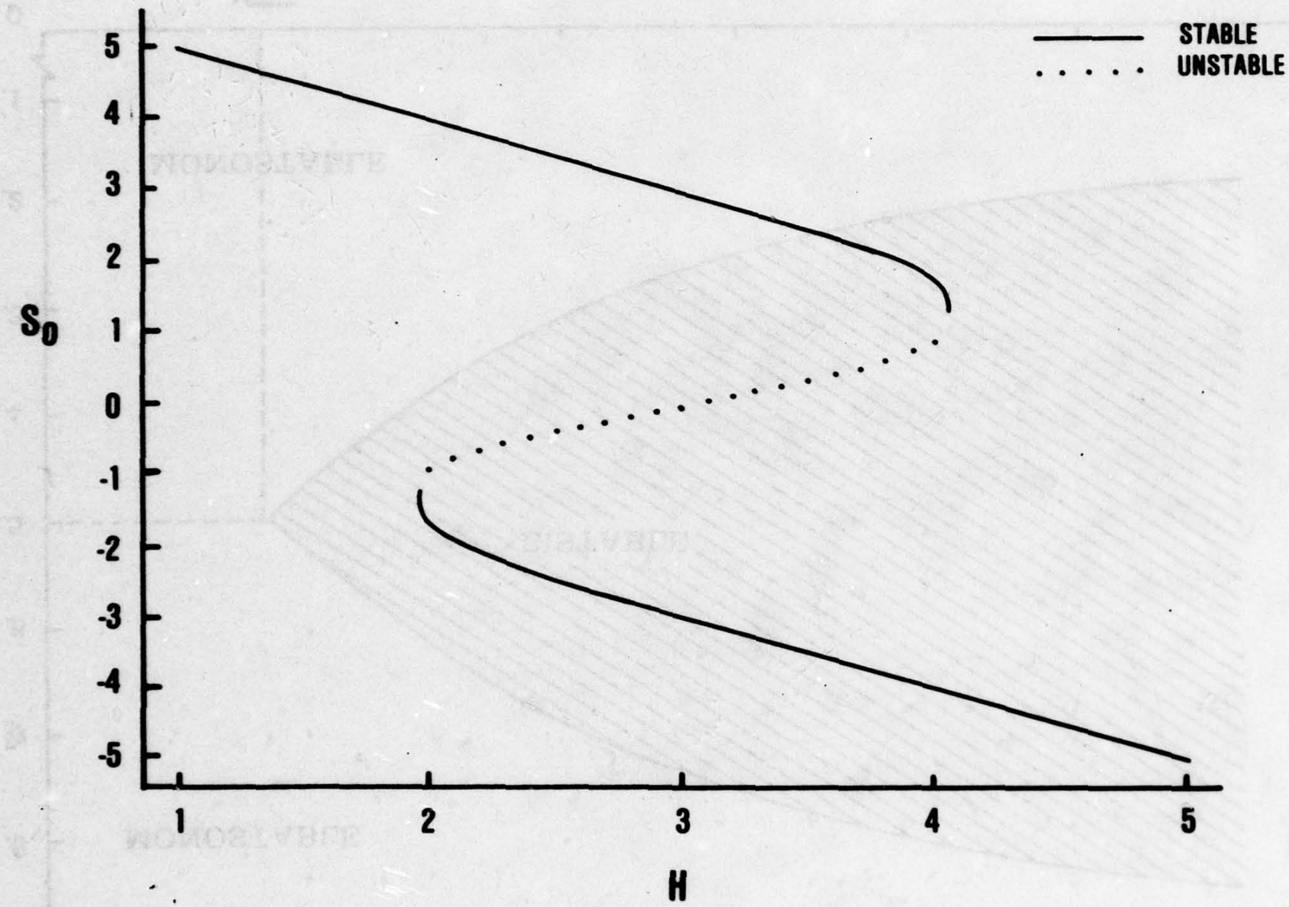

Figure 7. Equilibrium state(s) as a function of the threshold for a single Amari net. $W = 6.0$.

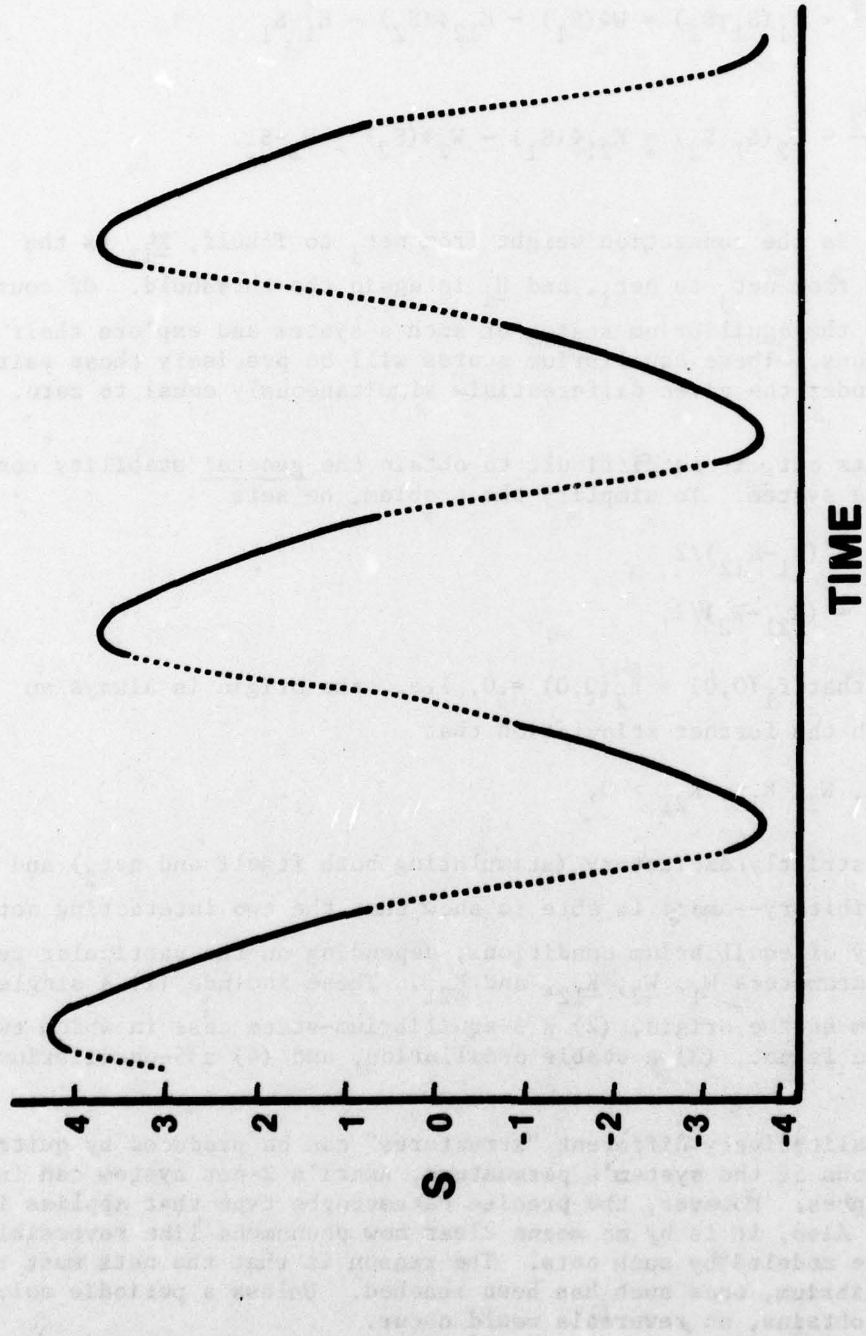

Figure 8. The catastrophic trajectory of a single Amari net having feedback to its threshold. S = the state variable; the abscissa corresponds to 500 iterations of the differential equation.

The Case of Two Interacting Nets

Amari (1972) also studied two interacting nerve nets, with respective activity levels S_1 and S_2, and used the following pair of differential equations:

$$\frac{dS_1}{dt} = f_1(S_1,S_2) = W\Phi(S_1) - K_{12}\Phi(S_2) - H_1 - S_1$$

$$\frac{dS_2}{dt} = f_2(S_1,S_2) = K_{21}\Phi(S_1) - W_2\Phi(S_2) - H_2 - S_2.$$

Here W_i $(i = 1,2)$ is the connection weight from net_i to itself, K_{ij} is the connection weight from net_j to net_i, and H_i is again the threshold. Of course, one wants to find the equilibrium states of such a system and explore their stability conditions. These equilibrium states will be precisely those pairs (S_1,S_2), which render the given differentials simultaneously equal to zero.

As Amari points out, it is difficult to obtain the general stability conditions for such a system. To simplify the problem, he sets

$$H_1 = (W_1 - K_{12})/2$$

$$H_2 = (K_{21} - W_2)/2,$$

which guarantees that $f_1(0,0) = f_2(0,0) = 0$, i.e., the origin is always an equilibrium. With the further stipulation that

$$W_1,\ W_2,\ K_{12},\ K_{21} > 0,$$

which makes net_1 strictly excitatory (stimulating both itself and net_2) and net_2 strictly inhibitory--Amari is able to show that the two interacting nets may have a variety of equilibrium conditions, depending on the particular relations among the parameters W_1, W_2, K_{12}, and K_{21}. These include (1) a single stable equilibrium at the origin, (2) a 3-equilibrium-state case in which two are stable and one is not, (3) a stable oscillation, and (4) a 5-equilibrium-state case.

Since such qualitatively different "structures" can be produced by quite simple manipulations of the system's parameters, Amari's 2-net system can indeed manifest catastrophes. However, the precise catastrophe type that applies is unknown (to us). Also, it is by no means clear how phenomena like reversible perceptions can be modeled by such nets. The reason is that the nets must remain at a stable equilibrium, once such has been reached. Unless a periodic solution (an oscillation) obtains, no reversals would occur.

We next made several modifications to Amari's system. A most simple change was to let $\underline{K}_{21} < 0 < \underline{K}_{12}$, so that the nets become mutually inhibitory, which seems intuitively more likely to model the phenomena of reversible perception. Also, we took $\underline{W}_2 < 0 < \underline{W}_1$ so that both nets would be self-stimulating.

The PDP-12 computer was used to solve for the resulting stability conditions. Surprisingly, such a pair of self-exciting, mutually inhibitory nets can manifest as many as nine distinct equilibria. For instance, Figure 9 is a plot in the (S_1,S_2) plane and shows the nine obtained equilibria as the <u>intersections</u> of the solution functions for f_1 and f_2 separately. Here $\underline{W}_1 = 6.0$, $\underline{W}_2 = -6.0$, $\underline{K}_{12} = 2.0$, and $\underline{K}_{21} = -2.0$. It turns out that the four equilibria indicated by the filled circles in Figure 9 are stable; the other five equilibria are unstable. These results exhibit the complexity and versatility of such simple hypothetical nets; unfortunately, no ready insight is gained with respect to their implications as models of perceptual phenomena.

We next introduced feedback into the basic Amari equations in a manner somewhat related to that used by Zeeman (1973, p. 706) in modeling the heartbeat. Here we postulated an habituation/recovery effect that raises/lowers the threshold for each net. In particular, each threshold $\underline{H}_i$ has its own differential equal to S_i/D where D is again a damping constant. Clearly, for $\underline{S}_i > 0$ the threshold of net_i is raised (habituation), and for $\underline{S}_i < 0$ the threshold of net_i is lowered (recovery).

Digital simulations were implemented on the PDP-12. With these final modifications, the behavior of the mutually inhibitory nets is cyclic. Figure 10 is a sample trajectory in the (S_1,S_2) plane, where $\underline{W}_1 = 6.0$, $\underline{W}_2 = -6.0$, $\underline{K}_{12} = 3.0$, and $\underline{K}_{21} = -3.0$. The initial values of $\underline{H}_1$, $\underline{H}_2$, $\underline{S}_1$, and $\underline{S}_2$ were 1.0, 1.5, 1.0, and -1.0 respectively. The damping constants for the $\underline{S}_i$'s and $\underline{H}_i$'s were 20 and 40 respectively. It can be seen in Figure 10 that, after an early transient, the system settles down to a cyclic walk along the line $S_2 = -S_1$.

A cautious claim is made for a "rough" qualitative fit between the behavior of such nets and the phenomena of reversible perception. However, we have negated, through the feedback loop, Amari's stipulation that $\underline{H}_1$ and $\underline{H}_2$ are fixed functions of $\underline{W}_1$, $\underline{W}_2$, $\underline{K}_{12}$, and $\underline{K}_{21}$. His results on the system's equilibrium conditions clearly no longer hold.

Accordingly, we cannot conclude that our cyclic nets manifest catastrophes and if so, of what kind. It is possible that the feedback on the thresholds sometimes brings the system within the range of a different, pre-existing attractor—and at other times creates new attractors and/or destroys old ones. Our own limited analytic techniques simply do not permit us to resolve these issues. However, these "simple" but conceptually rich mathematical nets, together with computer simulation techniques, do offer promise for the progressively better modeling of reversible processes. Catastrophe theory, at the very least, provides a context within which the behavior of the models can be better understood.

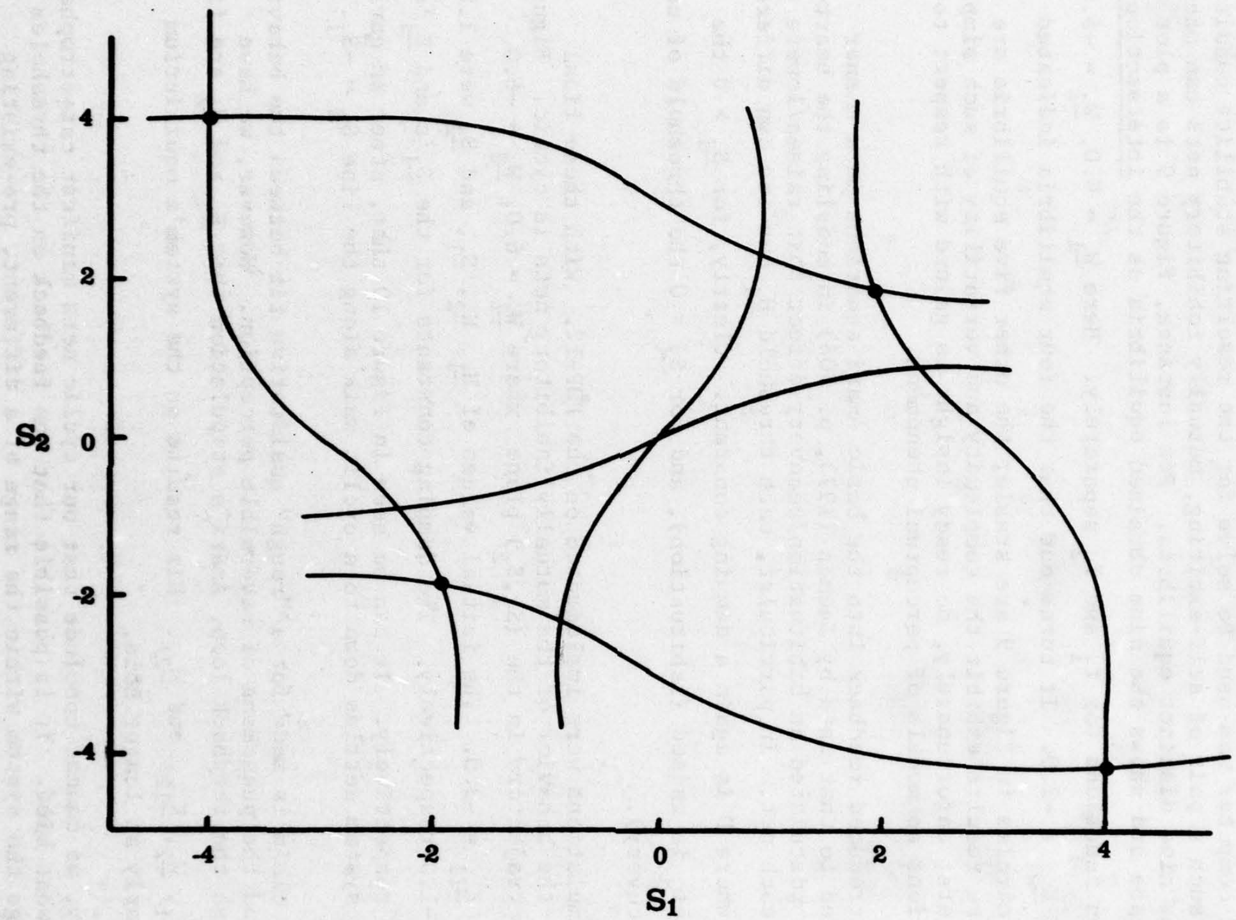

Figure 9. Equilibrium solutions, shown as intersections of the curves, for a pair of self-exciting, mutually inhibitory nets. The four circled points are stable equilibria.

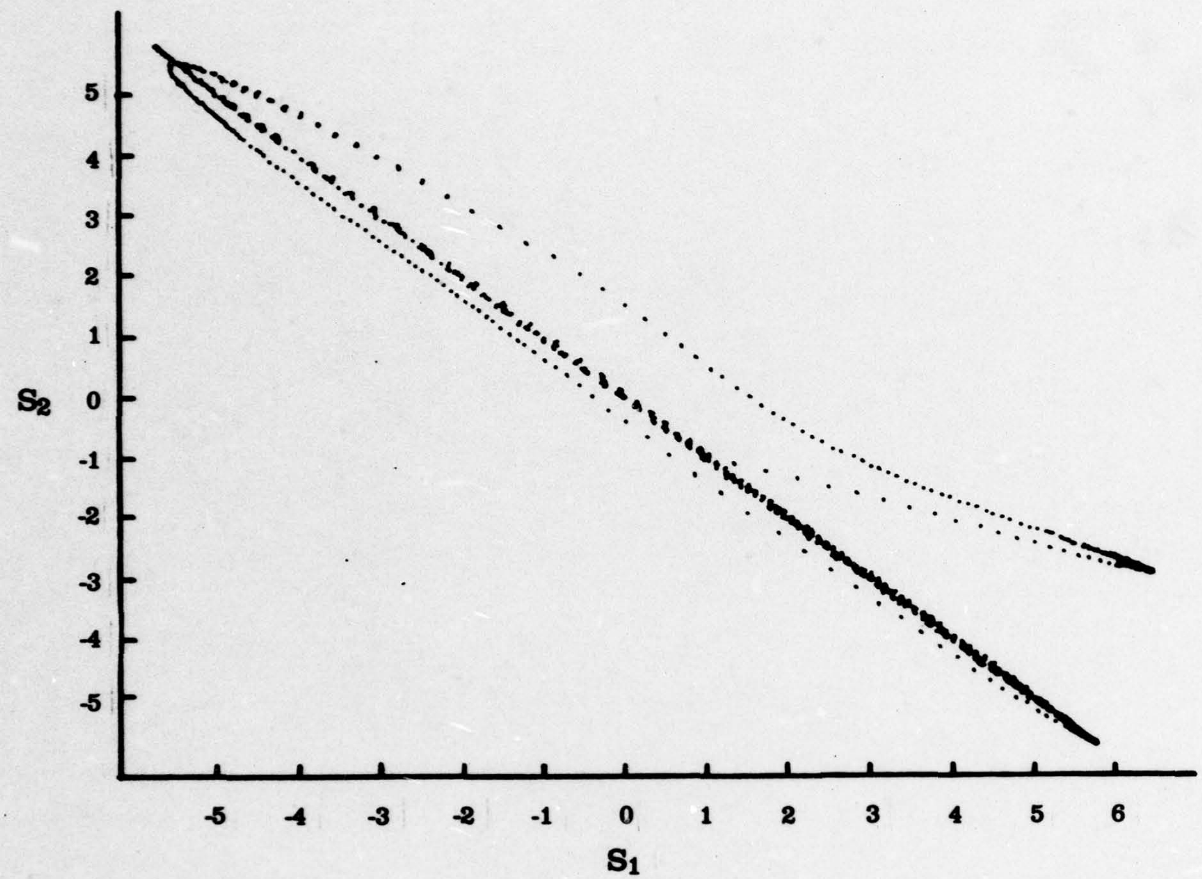

Figure 10. Trajectory for two mutually inhibitory nets with feedback.

DISCUSSION

Stability, Instability, and Past History

As we have seen, catastrophe theory is a study of the changing conditions of stability, the hallmark of which is the sudden appearance or disappearance of an equilibrium. The state of the system under study, however, may or may not change radically when such an equilibrium appears or disappears; that will depend upon whether or not the current state is such that it moves within the "sphere of influence" of the new "attractor." Thus, the manifestation of overt catastrophes depends jointly upon the current "topography" of the system's attractors and the system's current state. Note that the current state of the system, by hypothesis of a fast dynamic, is always close to a stable equilibrium.

Since this stable equilibrium in turn depends on the past history of the system's behavior, previous history emerges as a critical factor bearing on the impact of changes in the values of control variables. We see this clearly in Zeeman's catastrophe machine. Thus, when the values of the control variables move from inside the bifurcation set to the outside, the system may either continue to rest at nearly the same position or leap to a new energy minimum. The result depends upon which minimum "held" the machine when the attractor structure changed.

Some of our observations on apparent movement showed very clearly this dependence upon immediate past history. Consider the contingent procedure in which subjects responded first to the square stimulus. For some nearby stimulus values, vertical motion was always or nearly always reported if the immediately preceding perception was vertical. Yet stimuli that were objectively more vertical were seen as horizontal if the immediately preceding perception had been of horizontal movement. This powerful effect of sequence is precisely in keeping with the assumptions of catastrophe theory.

Implications for Conduct of Psychological Experiments

Sequential effects have, of course, been very clearly recognized by psychological methodologists. However, the <u>attitude</u> toward these effects has generally been quite different from that which derives from a study of catastrophe theory. Psychological statisticians largely regard these effects as nuisances to be minimized; thus, Latin and Greco-Latin squares are used so that sequential effects can be partially randomized and balanced (Winer, 1971). Even so, statisticians recognize that these procedures are only partially successful in eliminating possible sequential effects, and they frequently issue cautions about limitations on generalizing as to the possible operation of order effects. Very rarely are order effects suggested as topics of study in themselves. Winer has more to say about sequential effects than most of the writers of popular psychological statistics texts, and the following quote seems to capture his attitude best (1971):

> . . . in cases where the sequence of administration of the treatments was not dictated by the nature of the experimental variables, it was suggested that order be randomized independently for each subject. A partial control of sequence

> effects is provided by the use of the Latin square principle . . . A more complete control of sequence effects (but one which is more costly in terms of experimental effort) is available . . . [One may] select deliberately representative sequences from among the total possible sequences. (p. 576)

Winer's statement appears to exemplify the view of sequential processes as largely nuisances, and to recommend a systematic order of presentation (low to high or high to low) only when such an order is "dictated by the nature of the experimental variables." Note that this will be the case if the experimental variable is time or trials, which present themselves only in increasing order. The catastrophe theorist would suggest that such an order is desirable in any event, and that one is doubly fortunate if both increasing and decreasing orders are possible.

A consideration of catastrophe theory leads us to see sequential effects, particularly the hysteresis effects they may reveal, as fundamentally important and interesting phenomena. Only our interest in catastrophe theory led us to search for and find the strong and rather peculiar order effects in the apparent movement experiment. We do not claim that our discovery of these kinds of effects was in any way unique or original; the historical interest in the phenomena of set and adaptation level belies such a claim. We do believe that the philosophy underlying catastrophe theory provides a framework that unites the various types of order phenomena, and a language that helps investigators discuss and understand them. Thus, an understanding of the types of problems with which catastrophe theory deals might lead experimenters to take a new attitude toward the conduct of psychological experiments. Surely there will continue to be a place for experimentation in which values of independent variables are presented in random order, and attempts are made to eliminate order effects. There is also a need, however, for experiments in which many values of independent variables are presented in systematic, ordered sequences.

Croll (1976) in the midst of an article highly critical of catastrophe theory, makes another methodological point:

> If one is observing an inherently nonlinear process, such as the highly optimized and parametrically inter-connected processes of the "soft sciences," then no amount of linear correlation between the statistical parameters will yield meaningful results. Or, in the language of the catastrophist, "no approach which uses any kind of statistical averaging will locate a multi-sheeted graph, even if it actually occurs, because each layer of sheets will be replaced by a single sheet in the average position." If catastrophe theory has contributed to reinforcing this message, it will have performed a useful function. (p. 632)

Clearly much more empirical data are needed before one can resolve fully questions about the usefulness of catastrophe theory in the realm of behavioral science. Meanwhile, we can but agree with Thom (1976): "But as in the 'soft sciences' we have nothing better, why not try?"

CONCLUSIONS

Analysis of the purported applications of catastrophe theory to behavior showed that they failed reasonable tests of rigor. Three new applications were devised and tested in the laboratory, but these yielded only partial success. Simulations of nerve nets manifested catastrophes and showed some promise for modeling perceptual phenomena. There is as yet no convincing demonstration, however, of the theory's application to behavioral phenomena. The controversy surrounding catastrophe theory stems from its alleged applications; there is no indication of its falsity as a mathematical result. Catastrophe theory is conceptually rich and provides a new way of looking at and conducting psychological experiments. While its promise has not yet been realized, it is far too early to dismiss the theory.

RECOMMENDATIONS

Efforts should continue toward finding adequate behavioral applications of catastrophe theory. Perceptual phenomena are deemed very promising, but investigation should extend to other realms as well.

Work should proceed toward further developing statistical procedures that will help in evaluating the fit of catastrophe theory models to data. Techniques are especially needed to compare the fit of catastrophe models with that of competing theories.

The implications of catastrophe theory for general experimental methodology should be further examined. Renewed emphasis should be placed on the sequential and systematic manipulation of independent variables as a tool in behavioral methodology.

The dynamics of neural systems, including mutually inhibitory systems, should be further explored through analysis and simulation. One such simulated system has been shown to exhibit the cusp catastrophe and provides a possible first basis for the eventual proper union of catastrophe theory with behavioral science.

REFERENCES

Amari, S-I. Characteristics of random nets of analog neuron-like elements. IEEE Transactions on Systems, Man, and Cybernetics, 1972, SMC-2, 643-657.

Amari, S-I., Yoshida, K., & Kanatani, K-I. A mathematical foundation for statistical neurodynamics. SIAM Journal of Applied Mathematics, 1977, 33, 95-126.

Attneave, F. Multistability in perception. Scientific American, 1971, 225(6), 62-71.

Buckner, D. N., & McGrath, J. J. (Eds.). Vigilance: A symposium. New York: McGraw-Hill, 1963.

Croll, J. Is catastrophe theory dangerous? New Scientist, 17 June 1976, pp. 630-632.

Davies, D. R., & Tune, G. S. Human vigilance performance. New York: American Elsevier, 1969.

Frey, P. W., & Sears, R. J. Model of conditioning incorporating the Rescorla-Wagner associative axiom, a dynamic attention process, and a catastrophe rule. Psychological Review, 1978, 85, 321-340.

Julesz, B. Foundations of cyclopean perception. Chicago: University of Chicago Press, 1971.

Kolata, G. B. Catastrophe theory: The emperor has no clothes. Science, 15 April 1977, p. 287.

Lewis, M. Letters. Science, 17 June 1977, p. 1270.

Poston, T., & Woodcock, A. E. R. Zeeman's catastrophe machine. Proceedings of the Cambridge Philosophical Society, 1973, 74, 217-226.

Price, J. R. Studies of reversible perspective: A methodological review. Behavioral Research Methods and Instrumentation, 1969, 1, 102-106.

Rashevsky, N. Mathematical biophysics. Chicago: University of Chicago Press, 1948.

Steen, L. A. Catastrophe theory: The first decade. Science News, 2 April 1977, p. 218.

Taylor, M. M., & Aldridge, K. D. Stochastic processes in reversing figure perception. Perception and Psychophysics, 1974, 16, 9-27.

Thom, R. [Structural stability and morphogenesis.] (D. H. Fowler, trans.). Reading, MA: W. A. Benjamin, 1975. (Originally published, 1972.)

Thom, R. (A reply to Dr. Croll). New Scientist, 17 June 1976, p. 632.

Thompson, D'A. On growth and form. London: Cambridge University Press, 1961.

Thompson, J. M. T. Instabilities, bifurcations, and catastrophes. Physics Letters, 1975, 51A(4), 201-203.

Thompson, J. M. T., & Hunt, G. W. Toward a united bifurcation theory. Journal of Applied Mathematics and Physics, 1975, 26, 581-603.

von Bekesy, G. Sensory inhibition. Princeton: Princeton University Press, 1967.

Winer, B. J. Statistical principles in experimental design. New York: McGraw-Hill, 1971.

Zeeman, E. C. Differential equations for the heartbeat and nerve impluse. In M. M. Peixoto (Ed.). Dynamical systems. New York: Academic Press, 1973.

Zeeman, E. C. Levels of structure in catastrophe theory illustrated by applications in the social and biological sciences. Proceedings of the International Congress of Mathematicians, 1974, Vancouver, 533-546.

Zeeman, E. C. Catastrophe theory. Scientific American, 1976, 234(4), 65-83. (a)

Zeeman, E. C. A mathematical model for conflicting judgements caused by stress applied to possible misestimations of speed caused by alcohol. British Journal of Mathematical and Statistical Psychology, 1976, 29, 19-31. (b)

Zeeman, E. C., Hall, C. S., Harrison, P. J., Marriage, G. H., & Shapland, P. H. A model for institutional disturbances. British Journal of Mathematical and Statistical Psychology, 1976, 29, 66-80.

REFERENCE NOTE

1. Hillix, W. A., & Hershman, R. L. Catastrophe theory, response rules, and nerve nets: A comment on Frey and Sears. Unpublished manuscript, 1978. Available from Navy Personnel Research and Development Center, (Code 305), San Diego, CA 92152.

BIBLIOGRAPHY

BIBLIOGRAPHY

Abraham, R., & Marsden, J. E. Foundations of mechanics. New York: Benjamin, 1967.

A'Campo, N. Le groupe de monodromie du deploiuement des singularites isolees de courbes planes I. Mathematische Annalen, 1975, 213, 1-32.

Aggarval, J. K. Notes on nonlinear systems. New York: van Nostrand Reinhold, 1972.

Araki, J., Yano, T., Ueda, M., & Noda, M. T. Spontaneously broken symmetry and the cusp catastrophe. Proceedings of the Royal Society of London, 1975, A-345, 413-416.

Amari, S-I. Characteristics of randomly connected threshold-element networks and network systems. Proceedings of the IEEE, 1971, 59, 35-47.

Amari, S-I. Characteristics of random nets of analog neuron-like elements. IEEE Transactions on Systems, Man, and Cybernetics, 1972, SMC-2, 643-657.

Amari, S-I. A mathematical approach to neural systems. In J. Metzler (Ed.), Systems neuroscience. New York: Academic Press, 1977.

Amari, S-I, Yoshida, K., & Kanatani, K-I. A mathematical foundation for statistical neurodynamics. SIAM Journal of Applied Mathematics, 1977, 33, 95-126.

Arnold, V. I. Proceedings of the International Congress of Mathematicians, 1974, Vancouver, 19-39.

Atiyah, M. F. International colloquium on differential analysis (Bombay, 1964). London: Oxford University Press, 1964.

Baker, J. S. The bimodality and hysteresis properties of response mapping rules based on the cusp catastrophe. Evanston, IL: Unpublished doctoral dissertation, Northwestern University, 1978.

Balasko, Y. The behavior of economic equilibria: A catastrophe theory approach. Behavioral Science, 1978, 23, 375-382.

Banchoff, T. Polyhedral catastrophe theory: I. Maps of the line to the line. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Benham, C. J., & Kozak, J. J. Denaturation: An example of a catastrophe: II. Two state transitions. Journal of Theoretical Biology, 1976, 63, 125-149.

Benham, C. J., & Kozak, J. J. Denaturation: An example of a catastrophe: III. Phase diagrams for multistable transformations. Journal of Theoretical Biology, 1977, 66, 681-693.

Benham, C. J., & Kozak, J. J. Catastrophes in statistical biophysics. Behavioral Science, 1978, 23, 355-359.

Berlinski, D. Catastrophe theory and its applications: A critical review. Behavioral Science, 1978, 23, 402-416.

Berry, M. V. Cusped rainbows and incoherence effects in the rippling-mirror model for particle scattering from surfaces. Journal of Physics A, 1975, 8(4), 566-584.

Boardman, J. M. Singularities of differentiable maps. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1967, 33, 393-419.

Bowers, R. G. Statistical dynamic models of social systems: Discontinuity and conflict. Behavioral Science, 1978, 23, 120-129.

Brocker, T. [Differentiable germs and catastrophes.] (L. Lander, trans.). London: Cambridge University Press, 1976.

Bryant, H. C., & Jaimie, N. The glory. Scientific American, 1975, 231, 60-71.

Callahan, J. Singularities and plane maps. American Mathematical Monthly, 1974, 81(3), 211-240.

Chern, S. S., & Smale, S. (Eds.). Global analysis. Providence: American Mathematical Society, 1970.

Chernoff, P. R., & Marsden, J. E. Properties of infinite dimensional Hamiltonian systems. Lecture notes in Mathematics, 425, Berlin: Springer-Verlag, 1974.

Chidley, J., Lewis, P., & Walker, P. The cusp catastrophe as a market planning aid. Behavioral Science, 1978, 23, 351-354.

Chillingworth, D. (Ed.). Proceedings of the Symposium on Differential Equations and Dynamical Systems. Lecture notes in Mathematics, 206, Berlin: Springer-Verlag, 1971.

Chilver, H. Wider implications of catastrophe theory. Nature, 1975, 254, 381.

Clarke, B. L. Graph theoretic approach to the stability analyses of steady state chemical reaction networks. Journal of Chemical Physics, 1974, 60(4), 1481-1492.

Cobb, L. Stochastic catastrophe models and multimodal distributions. Behavioral Science, 1978, 23, 360-374.

Conrad, M. Statistical and hierarchical aspects of biological organizations. In C. H. Waddington (Ed.), Toward a theoretical biology, (Vol. IV). Chicago: Aldine, 1972.

Croll, J. Is catastrophe theory dangerous? New Scientist, 17 June 1976, pp. 630-632.

Cronin, J. Some mathematics of biological oscillations. SIAM Review, 1977, 19, 100-138.

Deakin, M. A. B. Catastrophe theory and its applications. The Mathematical Scientist, 1977, 2, 73-94.

DeGreene, K. B. Force fields and emergent phenomena in sociotechnical macro-systems: Theories and models. Behavioral Science, 1978, 23, 1-14.

Delattre, P. Formalization concepts and exploration concepts. Scientia, 1974, 109(5-8), 459-481.

Dodson, M. M. Darwin's law of natural selection and Thom's theory of catastrophes. Mathematical Bioscience, 1976, 28, 243-274.

Dubois, J., & Dufour, J. La theorie des catastrophes: I. La machine a catastrophes. II. Dynamiques gradientes a une variable d'etat. Annales Institut Henri Poincare, 1974, 20(2), 113-115.

DuFour, J. Une limite aux extensions du theoreme de preparation. Comptes Rendus Academie des Sciences Paris, 1976A, 282, 199-202.

Ellis, R. Lectures on topological dynamics. New York: Benjamin, 1969.

Engasser, J-M., Flamm, M., & Horvath, C. Hormone regulation of cellular metabolism: Interplay of membrane transport and consecutive enzyme reaction. Journal of Theoretical Biology, 1977, 67, 433-445.

Frankhauser, H. R. Phasenubergange als Kalastrophen ein Beispiel. Helvetica Physica Acta, 1974, 47(4), 486-490.

Fararo, T. J. An introduction to catastrophes. Behavioral Science, 1978, 23, 291-317.

Field, M. Transveralite dans les G-varietes. Comptes Rendus Academie des Sciences Paris, 1976A, 282, 115-117.

Field, M. Une version equivariante du theoreme de la transversalite de Thom. Comptes Rendus Academie des Sciences Paris, 1976A, 282, 379-380.

Flay, B. R. Catastrophe theory in social psychology: Some applications to attitudes and social behavior. Behavioral Science, 1978, 23, 335-350.

Forster, W. Katastrophentheorie. Acta Physica Austriaca, 1974, 39, 201-211.

Fowler, D. W. The Riemann-Hugoniot catastrophe and van der Waals Equations. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Frey, P. W., & Sears, R. J. Model of conditioning incorporating the Rescorla-Wagner associative axiom, a dynamic attention process, and a catastrophe rule. Psychological Review, 1978, 85, 321-340.

Fuchs, G. Structural stability for dynamical economics models. Journal of Mathematical Economics, 1975, 2, 139-154.

Furutani, N. A new approach to traffic behavior: I. Modelling of "following-defence" behavior. International Journal of Man-Machine Studies, 1976, 8, 597-615.

Furutani, N. A new approach to traffic behavior: II. Individual car and traffic flow. International Journal of Man-Machine Studies, 1976, 8, 731-742.

Furutani, N. A new approach to traffic behavior: III. Steering behavior and the butterfly catastrophe. International Journal of Man-Machine Studies, 1977, 9, 233-254.

Garstens, M. A. Remarks on statistical mechanics and theoretical biology. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. III). Chicago: Aldine, 1970.

Glasner, S. Topological dynamics and group theory. Transactions of the American Mathematical Society, 1974, 187(1), 327-334.

Godwin, A. N. Three dimensional pictures for Thom's parabolic umbilic. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1971, 40, 117-138.

Godwin, A. N. Toplogical bifurcation for the double cusp polynomial. Mathematical Proceedings of the Cambridge Philosophical Society, 1975, 77, 293-312.

Golubitsky, M. An introduction to catastrophe theory and its applications. SIAM Review, 1978, 20, 352-387.

Goodwin, B. C. Biological stability. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol. III). Chicago: Aldine, 1970.

Goodwin, B. C. The division of cells and the fusion of ideas. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Gorman, J. The shape of change. The Sciences, September 1976, pp. 17-22.

Gottinger, H. Complexity and dynamics: Applications of dynamic systems theory. IEEE Transactions on Systems, Man, and Cybernetics, 1976, SMC-6, 867-873.

Griffith, J. S. A view of the brain. London: Oxford University Press, 1967.

Griffiths, R. B. Phase diagrams and higher-order critical points. Physical Review B, 1975, 12(1), 345-355.

Gromoll, D., & Meyer, W. On differentiable functions with isolated critical points. Topology, 1969, 8, 361-369.

Guckenheimer, J. Review of 'Stabilite Structurelle et Morphogenese' by Rene Thom. Santa Cruz: Author, 1973.

Guckenheimer, J. Catastrophes and partial differential equations. Annales Institut Fourier (Grenoble), 1973, 23(2), 31-59.

Guckenheimer, J. Bifurcation and catastrophe. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Guckenheimer, J. One-parameter families of vector fields on two-manifolds: Another nondensity theorem. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Guckenheimer, J. Caustics and non-degenerate Hamiltonians. Topology, 1974, 13, 127-133.

Guillemin, V., & Pollak, A. Differential topology. Englewood Cliffs, NJ: Prentice-Hall, 1974.

Gurel, O. Topological dynamics in neurobiology. International Journal of Neuroscience, 1973, 6, 165-179.

Guyon, E. Second-order phase transitions: Models and analogies. American Journal of Physics, 1973, 43(10), 877-881.

Hale, J. K. Some infinite-dimensional dynamical systems. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Hawking, S. W. The definition and occurrence of singularities in general relativity. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium II. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Hawking, S. W., King, A. R., & McCarthy, P. J. A new topology for curved space-time which incorporates the causal, differential, and conformal structures. *Journal of Mathematical Physics*, 1976, 17(2), 174-181.

Hermann, R. Vector bundles in mathematical physics (Vols. I & II). New York: Benjamin, 1970.

Hermann, R. Geometry, physics, and systems. New York: Marcel Dekker, 1973.

Hilton, P. Structural stability, the theory of catastrophes, and applications in the sciences. Lecture notes in Mathematics, 525, Berlin: Springer-Verlag, 1976.

Hirsch, M. W., & Smale, S. Differential equations, dynamical systems, and linear algebra. New York: Academic Press, 1974.

Isnard, C. A., & Zeeman, E. C. Some models from catastrophe theory in the social sciences. In L. Collins (Ed.), The use of models in the social sciences. London: Tavistock Publications, 1976.

Janich, K. Caustics and catastrophes. Mathematische Annalen, 1974, 209, 161-180.

Jones, D. D., & Walters, C. J. Catastrophe theory and fisheries regulation. Journal of the Fisheries Research Board of Canada, 1976, 33, 2829-2833.

Keener, J. P. Perturbed bifurcation theory at multiple eigenvalues. Archive for Rational Mechanics and Analysis, 1974, 56(4), 348-366.

Keller, J. B., & Antman, S. (Eds.). Bifurcation theory and nonlinear eigenvalue problems. New York: Benjamin, 1969.

Klingenberg, W. Closed geodesics. Annals of Mathematics, 1969, 89, 68-91.

Kolata, G. B. Cascading bifurcations: The mathematics of chaos. Science, 1974, 189, 984.

Kolata, G. B. Catastrophe theory: The emperor has no clothes. Research News, Science, 15 April 1977, p. 287.

Kozak, J. J., & Benham, C. J. Denaturation: An example of a catastrophe. Proceedings of the National Academy of Sciences, U.S.A., 1974, 71, 1977-1981.

Kuo, T-C. The jet space $J^n(n,p)$. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium I. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Kupka, I. On two notions of structural stability. Journal of Differential Geometry, 1974, 9, 639-644.

Lacher, R. C., McArthur, R., & Buzyna, G. Catastrophic changes in circulation flow patterns. American Scientist, 1977, 65, 614-621.

Lander, L. The structure of the Thom-Boardman singularities of stable germs with type $\Sigma^{2,0}$. Proceedings of the London Mathematical Society, 1976, 33(3), 113-137.

Leigh, E. G., Jr. The ecological role of Volterra's equation. In M. Gerstenhaber (Ed.), Some mathematical problems in biology. Lectures on mathematics in life sciences, Providence: American Mathematical Society, 1968.

Levine, H. I. Singularities of differentiable mappings. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium I. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Levine, H. I. Blowing up singularities. In C. T. C. Wall (Ed.)., Proceedings of the Liverpool Singularities Symposium II. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Levins, R. Complex systems. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. III). Chicago: Aldine, 1970.

Levins, R. Extinction. In M. Gerstenhaber (Ed.), Some mathematical questions in biology. Lectures on mathematics in the life sciences (Vol. 2), Providence: American Mathematical Society, 1970.

Lu, Y-C. Singularity theory and an introduction to catastrophe theory. New York: Springer-Verlag Universitext, 1976.

MacArthur, R. Graphical analyses of ecological systems. In M. Gerstenhaber (Ed.), Some mathematical questions in biology. Lectures on mathematics in the life sciences (Vol. 2), Providence: American Mathematical Society, 1970.

Magnus, R. J. On universal unfoldings of certain real functions on a Banach space. Mathematical Proceedings of the Cambridge Philosophical Society, 1977, 81, 91-97.

Marion, J. B. Classical dynamics of particles and systems. New York: Academic Press, 1965.

Markus, L. Controlability in topological dynamics. Proceedings of the International Congress of Mathematicians, 1974, Vancouver, 355-360.

Marsden, J. Applications of global analyses in mathematical physics. Boston: Publish or Perish, 1974.

Mather, J. N. Stability of C^∞ mappings: I. The Division Theorem. Annals of Mathematics, 1968, 89, 89-104.

Mather, J. N. Stability of C^∞ mappings: III. Finitely determined map-germs. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1968, 127-156.

Mather, J. N. Stability of C^∞ mappings: II. Infinitesimal stability implies stability. Annals of Mathematics, 1969, 89, 254-291.

Mather, J. N. Stability of C^∞ mappings: IV. Classification of stable germs. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1970, 223-248.

Mather, J. N. Stability of C^∞ mappings: V. Transversality. Advances in Mathematics, 1970, 4, 301-336.

Mather, J. N. Stability of C^∞ mappings: VI. The Nice dimensions. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium I. Lecture notes in Mathematics, 191, Berlin: Springer-Verlag, 1971.

Mather, J. N. On Thom-Boardman singularities. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Mather, J. N. Solutions of generic linear equations. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Mather, J. N. Stratifications and mappings. In M. M. Peixoto (Ed.), Dynamical systems. New York, Academic Press, 1973.

Matsushima, Y. Differentiable manifolds. New York: Marcel Dekker, 1972.

May, R. M. Simple mathematical models with very complicated dynamics. Nature, 1976, 261, 459-467.

Mayr, E. Cause and effect in biology. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. I). Chicago: Aldine, 1968.

Menon, M. V. Are there applications of catastrophe theory? Review of a talk by Professor Hector Sussman at a Joint Mathematics and Physics Colloquium at the University of Chicago. Office of Naval Research Review, October-November 1976.

Morowitz, H. J. The derivation of ecological relationships from physical and chemical principles. Proceedings of the National Academy of Sciences, U.S.A., 1974, 71(6), 2335-2336.

Morse, M. The calculus of variations in the large. American Mathematical Society Colloquium Publications (Vol. XVIII). Providence: A.M.S., 1934.

Moser, J. Dynamical systems, theory, and applications. Berlin: Springer-Verlag, 1975.

Mount, K. R., & Villamayor, O. E. An algebraic construction of the generic singularities of Boardman-Thom. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1973, 205-244.

Newhouse, S., & Poles, J. Bifurcations of Morse-Smale dynamics systems. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Nicolis, G., & Auchmutz, J. F. G. Dissipative structures, catastrophes, pattern formation: A bifurcation analysis. Proceedings of the National Academy of Sciences, U.S.A., 1974, 71(7), 2748-2751.

Olive, D. Singularities in relativistic quantum mechanics. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium II. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Overington, W. J. G. Catastrophe theory and the analog computer. Transactions of the International Association for Mathematics and Computers in Simulations, 1976, 18.

Palais, R. S. Foundations of global non-linear analysis. New York: Benjamin, 1968.

Palais, R. S., & Smale, S. A generalized Morse theory. Bulletin of the American Mathematical Society, 1964, 70, 165-172.

Pattee, H. H. Laws and constraints, symbols and languages. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Peixoto, M. M. Structural stability in two-dimensional manifolds. Topology, 1962, 1, 101-120.

Peixoto, M. M. On an approximation theorem of Kupka and Smale. Journal of Differential Equations, 1966, 3, 214-227.

Peixoto, M. M. On bifurcations of dynamical systems. Proceedings of the International Congress of Mathematicians. Vancouver, 1974, 315-335.

Pimbley, G. H. Eigenfunction branches of nonlinear operators, and their bifurcations. Lecture notes in Mathematics, 104, Berlin: Springer-Verlag, 1969.

Plant, R. E. Crustacean cardiac pacemaker model--an analysis of the singular approximation. Mathematical Biosciences, 1977, 36, 149-171.

Poston, T., & Stewart, I. N. Taylor expansions and catastrophes. London: Putnam, 1976.

Poston, T., & Stewart, I. Nonlinear modeling of multistable perception. Behavioral Science, 1978, 23, 318-334.

Poston, T., & Stewart, I. Catastrophe theory and its applications. Belmont, CA: Fearon-Pitman, 1978.

Poston, T., & Woodcock, A. E. R. Zeeman's catastrophe machine. Proceedings of the Cambridge Philosophical Society, 1973, 74, 217-226.

Poston, T., & Woodcock, A. E. R. A geometrical study of the elementary catastrophes. Lecture notes in Mathematics, 373, Berlin: Springer-Verlag, 1974.

Rand, D. Thresholds in Pareto sets. Journal of Mathematical Economics, 1976, 3, 139-154.

Rashevsky, N. Mathematical biophysics. Chicago: University of Chicago Press, 1948.

Robbin, J. E. Topological conjugacy and structural stability for discrete dynamical systems. Bulletin of the American Mathematical Society, 1972, 78(6), 923-953.

Rosen, R. Note: Hysteresis in systems with a single steady state. Bulletin of Mathematical Biology, 1976, 38, 87-92.

Rosen, R. Optimality principles in biology. London: Butterworths, 1967.

Rosen, R. Dynamical systems theory in biology (Vol. 1). New York: Wiley-Interscience, 1970.

Rosen, R. Foundations of mathematical biology (Vols. I-III). New York: Academic Press, 1973.

Rosen, R. Book review: "Structural stability and morphogenisis," by Rene Thom. Bulletin of Mathematical Biology, 1977, 39, 629-632.

Rosenhead, J. Prison "catastrophe." New Scientist, 15 July 1976, p. 140.

Saari, D. G. Singularities of Newtonian gravitational systems. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Saari, D. G. A qualitative model for the dynamics of cognitive processes. Journal of Mathematical Psychology, 1977, 15, 145-168.

Salvioli, S. E. On the theory of geometric objects. Journal of Differential Geometry, 1972, 7, 257-278.

Schaeffer, D. G. A regularity theorem for conservation laws. Advances in Mathematics, 1973, 2, 368-386.

Schulman, L. S. Stable generation of simple forms. Journal of Theoretical Biology, 1976, 57, 453-468.

Scott, A. C. Neurodynamics: A critical survey. Journal of Mathematical Psychology, 1977, 15, 1-45.

Shub, M. Structurally stable diffeomorphisms are dense. Bulletin of the American Mathematical Society, 1972, 78(5), 817-818.

Simon, J. C. Recent progress to formal approach of pattern recognition and scene analysis. Pattern Recognition, 1975, 7, 117-124.

Smale, S. On gradient dynamical systems. Annals of Mathematics, 1961, 74(1), 199-206.

Smale, S. Morse theory and a non-linear generalization of the Dirichlet problem. Annals of Mathematics, 1964, 80, 382-396.

Smale, S. Structurally stable systems are not dense. American Journal of Mathematics, 1966, 88, 491-496.

Smale, S. Global stability in dynamical systems. In C. T. Tamm (Ed.), Lectures in modern analysis and applications I. Lecture notes in Mathematics, 103, Berlin: Springer-Verlag, 1969.

Smale, S. On the mathematical foundations of electrical circuit theory. Journal of Differential Geometry, 1972, 7, 193-210.

Smale, S. Personal perspectives on mathematics and mechanics. In S. A. Rice, K. F. Freed, & J. C. Light (Eds.), Statistical mechanics, new concepts, new problems, new applications. Chicago: University of Chicago Press, 1972.

Smale, S. Stability and isotopy in discrete dynamical systems. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Smale, S. Global analysis and economics VI: Geometric analysis of Pareto optima and price equilibria under classic hypotheses. Journal of Mathematical Economics, 1976, 3, 1-14.

Sotomayor, J. Structural stability and bifurcation theory. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Steen, L. A. Mathematicians hail new theory. Science News, 1974, 106, 106-107.

Steen, L. A. Catastrophe theory: The first decade. Science News, 2 April 1977, p. 218.

Stewart, I. The seven elementary catastrophes. New Scientist, 20 November 1975, p. 447.

Sussmann, H. J., & Zahler, R. S. A critique of applied catastrophe theory in the behavioral sciences. Behavioral Science, 1978, 23, 383-389.

Sussmann, H. J., & Zahler, R. S. Catastrophe theory as applied to the social and biological sciences: A critique. Synthese, 1978, 37, 117-216.

Takeus, F. Singularities of vector fields. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1967, 33, 47-100.

Takeus, F. On Zeeman's tolerance stability conjecture. In N. H. Kuiper (Ed.), Manifolds (Amsterdam, 1970). Lecture notes in Mathematics, 197, Berlin: Springer-Verlag, 1971.

Takeus, F. A nonstabilizable jet of a vector field. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Thissen, W., & de Mol, C. The agricultural and persistent pollution subsystems in World 3. IEEE Transactions on Systems, Man, and Cybernetics, 1978, SMC-8, 159-172.

Thom, R. Une theorie dynamique de la morphogenese. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol. I). Chicago: Aldine, 1968.

Thom, R. Ensembles et morphismes stratifies. Bulletin of the American Mathematical Society, 1969, 74, 240-284.

Thom, R. Topological models in biology. Topology, 1969, 8, 313-335.

Thom, R. Topologie et linquistique. In A. Haefliger & R. Narasimhan (Eds.), Essays on topology and related topics. Berlin: Springer-Verlag, 1970.

Thom, R. Topological models in biology. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol. III). Chicago: Aldine, 1970.

Thom, R. The bifurcation subset of a space of maps. In N. H. Kuiper (Ed.), Manifolds (Amsterdam, 1970). Lecture notes in Mathematics, 197, Berlin: Springer-Verlag, 1971.

Thom, R. A global dynamical scheme for vertebrate embryology. In J. Cowan (Ed.), Some mathematical questions in biology (Vol. IV). Providence: American Mathematical Society, 1971.

Thom, R. Structuralism and biology. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Thom, R. Phase transitions as catastrophes. In S. A. Rice, K. F. Freed, & J. C. Light (Eds.), Statistical mechanics, new concepts, new problems, new applications. Chicago: University of Chicago Press, 1972.

Thom, R. Language et catastrophes: Elements pour une semantiques topologique. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Thom, R. [Structural stability and morphogenesis] (D. H. Fowler, trans.). Reading, MA: Benjamin, 1975. (Originally published, 1972.)

Thom, R. Structural stability, catastrophe theory, and applied mathematics. SIAM Review, 1977, 19, 189-201.

Thompson, J. M. T. A general theory for the equilibrium and stability of discrete conservative systems. Journal of Applied Mathematics and Physics (ZAMP), 1969, 20, 797-846.

Thompson, J. M. T. Experiments in catastrophe. Nature, 1975, 254, 392-395.

Thompson, J. M. T. Instabilities, bifurcations, and catastrophes. Physics Letters, 1975, 51A(4), 201-203.

Thompson, J. M. T., & Gaspar, Z. A buckling model for the set of umbilic catastrophes. Mathematical Proceedings of the Cambridge Philosophical Society, 1977, 82, 497-507.

Thompson, J. M. T., & Hunt, G. W. Toward a unified bifurcation theory. Journal of Applied Mathematics and Physics (ZAMP), 1975, 26, 581-603.

Waddington, C. H. The basic ideas of biology. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol I). Chicago: Aldine, 1968.

Waddington, C. H. Concepts and theories of growth, development, differentiation, and morphogenesis. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol. III). Chicago: Aldine, 1970.

Waddington, C. H. Form and information. In C. H. Waddington (Ed.), Towards a theoretical biology (Vol IV). Chicago: Aldine, 1972.

Waddington, C. H. A catastrophe theory of evolutions. Annals of the New York Academy of Science, 1974, 231, 32-42.

Waddington, C. H. Stabilization in systems. Futures, 1977, 9, 139-146.

Wall, C. T. C. Lectures on C^∞: Stability and classification. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium II. Lecture notes in Mathematics, 192, Berlin: Springer-Verlag, 1971.

Wall, C. T. C. Reflections on gradient vector fields. In C. T. C. Wall (Ed.), Proceedings of the Liverpool Singularities Symposium II. Lecture notes in Mathematics, 209, Berlin: Springer-Verlag, 1971.

Wall, C. T. C. A remark on gradient dynamical systems. Bulletin of the London Mathematical Society, 1972, 165-166.

Wasserman, A. Morse theory for G-manifolds. Bulletin of the American Mathematical Society, 1965, 71, 384-388.

Wasserman, G. Stability of unfoldings. Lecture notes in Mathematics, 393, Berlin: Springer-Verlag, 1974.

Wasserman, G. Stability of unfoldings in space and time. Acta Mathematica, 1975, 135(1-2), 57-128.

Wasserman, G. Stability of caustics. Mathematische Annalen, 1975, 216, 43-50.

Whitney, H. On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane. Annals of Mathematics, 1955, 62(3), 374-410.

Williams, R. F. Expanding attractors. Institut des Hautes Etudes Scientifiques Publications Mathematiques, 1973, 169-203.

Winfree, A. T. Rotating chemical reactions. Scientific American, 1971, 230, 82-95.

Woodcock, A. E. R. Cellular differentiation and catastrophe theory. Annals of the New York Academy of Science, 1970, 231, 60-76.

Woodcock, A. E. R. Landscapes of change: Catastrophe theory and biological processes. Behavioral Science, 1978, 23, 390-401.

Woodcock, A. E. R., & Poston, T. A. Geometrical study of the elementary catastrophes. New York: Springer-Verlag, 1974.

Woodcock, A. E. R., & Poston, T. A higher catastrophe machine. Mathematical Proceedings of the Cambridge Philosophical Society, 1976, 79, 343-350.

Yamamura, N. A mathematical approach to spatial distribution and temporal succession in plant communities. Bulletin of Mathematical Biology, 1976, 38, 517-526.

Zeeman, E. C. Appendix: A catastrophe machine. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Zeeman, E. C. Differential equations for the heartbeat and nerve impulses. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol. IV). Chicago: Aldine, 1972.

Zeeman, E. C. Applications of catastrophe theory. Tokyo International Conference on Manifolds. Tokyo: University of Tokyo Press, 1973.

Zeeman, E. C. Catastrophe theory in brain modeling. International Journal of Neuroscience, 1973, 6, 39-41.

Zeeman, E. C. Differential equations for the heartbeat and nerve impulse. In M. M. Peixoto (Ed.), Dynamical systems. New York: Academic Press, 1973.

Zeeman, E. C. Levels of structure in catastrophe theory illustrated by applications in the social and biological sciences. Proceedings of the International Congress of Mathematicians, 1974, Vancouver, 533-546.

Zeeman, E. C. On the unstable behavior of stock exchanges. Journal of Mathematical Economics, 1974, 1, 39-49.

Zeeman, E. C. Catastrophe theory. Scientific American, 1976, 234(4), 65-83.

Zeeman, E. C. A mathematical model for conflicting judgments caused by stress, applied to possible misestimations of speed caused by alcohol. British Journal of Mathematical and Statistical Psychology, 1976, 29, 19-31.

Zeeman, E. C. Catastrophe theory: Selected papers, 1972-1977. Reading, MA: Addison-Wesley Advanced Book Program, 1977.

Zeeman, E. C., & Buneman, O. P. Tolerance spaces and the brain. In C. H. Waddington (Ed.), Toward a theoretical biology (Vol I). Chicago: Aldine, 1968.

Zeeman, E. C., Hall, C. S., Harrison, P. J., Marriage, G. H., & Shapland, P. H. A model for institutional disturbances. British Journal of Mathematical and Statistical Psychology, 1976, 29, 66-80.

Zimmermann, U., Pilwat, G., & Riemann, R. Dielectric breakdown of cell membranes. Biophysical Journal, 1974, 14, 881-899.

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Chief of Naval Research (Code 200), (Code 430), (Code 450) (4), (Code 455), (Code 458) (2)
Chief of Information
Director of Navy Laboratories
Chief of Naval Education and Training (005), (N-5)
Commander, Naval Military Personnel Command (NMPC-013C)
Commander, Naval Ocean Systems Center (Code 447), (Code 823)
Commander, Naval Weapons Center
Commanding Officer, Office of Naval Research, Branch Office, Boston
Commanding Officer, Office of Naval Research, Branch Office, Chicago
Commanding Officer, Office of Naval Research, Branch Office, Pasadena
Commanding Officer, Naval Research Laboratory (Code 2620)
Superintendent, Naval Academy
Superintendent, Naval Postgraduate School
Personnel Research Division, Air Force Human Rescurces Laboratory, Brooks Air Force Base
Technical Library, Air Force Human Resources Laboratory, Brooks Air Force Base
Human Engineering Division, 6570 Aerospace Medical Research Laboratory, Wright-Patterson Air Force Base
Program Manager, Life Sciences, Director, Air Force Office of Scientific Research
Assistant Secretary of the Air Force (Research, Development, and Logistics)
Deputy Chief of Staff, Research, Development, and Acquisition, U. S. Air Force
Army Research Institute for the Behavioral and Social Sciences
Director, Environmental and Life Sciences, Office of the Under Secretary of Defense for Research and Engineering
Commandant, Industrial College of the Armed Forces
Director, Defense Advanced Research Projects Agency
Defense Documentation Center (12)